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The symmetry of convective transitions in space and time

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There is a close relationship between the symmetry changes in the plane and space groups of convective systems and the type of the bifurcation. To explore this relationship the plane- and space-group symmetry of many convective circulations is first classified using standard crystallographic notation. The transitions that occur between these patterns can be described by the loss of either a translational or a point-group symmetry element. These are referred to as klassengleiche and translationengleiche or k and t, transitions respectively. Any transition can be decomposed into a series of k and t transitions. The symmetry of the governing differential equations is most easily discussed when these are written in terms of potentials, and allows transitions to be classified as pitchfork, transcritical or Hopf bifurcations. Such classification can be carried out from symmetry alone, without any consideration of the functional form of the solutions, the Rayleigh number or the importance of the nonlinear terms. For this purpose it is convenient to define a factor group, the irreducible representations of which transform as do the variables and differential operators in the equations.

Analysis of planform transitions observed in convective flows when the viscosity is temperature dependent using plane groups shows that all except the transition from the conductive solution to hexagons are pitchfork bifurcations. The factor groups involved are the cyclic group Z_2, and the dihedral groups D_3 and D_5. When the viscosity is constant, space groups are needed, and symmetry arguments show that all except the Eckhaus instability are pitchfork bifurcations, including that from the conductive solution. Hopf transitions to solutions periodic in time, in double-diffusive, in small- and in large-Prandtl-number convection, all involve loss of reflection symmetry in time and the factor group is D_2. The same approach suggests how transitions to circulations that are not periodic in either space or time may occur by period doubling in space or in time or in both.

1. Introduction

Planform transitions between spatially periodic patterns in convecting systems obviously involve changes in symmetry. Since Wigner's (1959) classical work, to a physicist symmetry changes imply that group theory should be applied. The theory of space groups is particularly concerned with the symmetry of periodic lattices, and has been extensively studied by crystallographers. The object of this paper is to explore a variety of convective transitions from this point of view, using the necessary results from group theory that are outlined in §2.2. The principal result is an understanding of the nature of the loss of symmetry in the transitions observed. There is a close relationship between the change of symmetry and the nature of the bifurcation, which allows transitions to be classified as pitchfork, transcritical or Hopf bifurcations on the basis of the symmetry change alone. It is also important to emphasize what group theory cannot do. The method cannot determine the sign or the magnitude of the constants in the normal form equations, and therefore whether the transition is sub- or supercritical. Nor can it determine the critical Rayleigh number. Such questions must be answered by experiment, either in the laboratory or on a computer; or by analysis. Numerical experiments are rapidly becoming easier to perform, and the theory outlined below provides the framework for a systematic numerical investigation of transitions in space and time.

Previous applications of group theory to convective systems have mostly been concerned with marginal stability, and are rather abstract. Sattinger (1977, 1978) analysed the nature of the change in symmetry involved in the initial formation of a convective planform, and his work has been extended by Buzano & Golubitsky (1983), Golubitsky, Swift & Knobloch (1984), Golubitsky & Schaeffer (1985) and Golubitsky & Stewart (1985). These authors were all concerned with the onset of convection. The present study has two important differences from this earlier work. The governing equations are written in potential form, which results in a considerable simplification of the symmetry groups involved. The other difference involves the use of ideas from crystallography concerning the notation, classification, structure and interrelationships of line, plane and space groups. Crystallographers are also concerned with symmetry changes, e.g. in structural phase transitions, and have developed a group-theoretical approach that is essentially based on Landau theory. Some of these ideas can be applied to convective systems, though this must be done with care because the fluid-dynamical problem is never in thermodynamic equilibrium.

The first step in describing a transition is to determine the symmetry elements before and after the change occurs. It is convenient to consider the change to occur always from the more symmetric to the less symmetric system. The planform of steady circulations can be described by the plane groups. They are listed with their symmetry elements in Volume A of the *International Tables for Crystallography* (Hahn 1983), hereinafter referred to as IT (1983). If the circulation has three-dimensional symmetry elements, as do many constant-viscosity flows, then space groups are required that are also all listed in IT (1983). The Fourier series corresponding to all the plane and space groups are listed in the earlier *International Tables for X-ray Crystallography* (Henry & Lonsdale 1952, last revision 1977), which will be referred to as IT (1952). The elements of crystallographic groups are translations, reflections and rotations. Convective systems are described by the temperature difference with respect to some reference temperature and by two scalar potentials. The convective problem therefore allows the symmetry operation, $\mathbb{R}$, that reverses the sign of a scalar, as well as the usual proper and improper rotations and translations. Space groups that contain $\mathbb{R}$ in combination with some space-group operations are known as black and white groups and are not included in IT (1983). Crystallographers have worked with two- and three-dimensional space groups since the nineteenth century (see IT 1983) and with black and white space groups for more than fifty years (Heesch 1930, see Shubnikov & Koptsik 1974), but little use of this work seems yet to have been made in fluid dynamics.

The other difference between the space groups of IT (1983) and those expressed by convective circulations concerns time dependence, which leads to perodicity in time as well as space. The space groups required are then four-dimensional and also black and white. Though four-dimensional space groups have been studied (see Neubüser, Wondratschek & Bülow 1971; Bülow, Neubüser & Wondratschek 1971; Wondratschek, Bülow & Neubüser 1971; Brown *et al.* 1978), they have not attracted the same interest as have those in three dimensions.

One of the most fruitful approaches for crystallographic studies of phase transitions is Landau theory (see Landau & Lifshitz 1980, ch. 13), which starts from the requirement that the Gibbs free energy, or thermodynamic potential, Φ, corresponding to an equilibrium state, must be a minimum with respect to the order parameter. No corresponding extremum principles are known for the Navier–Stokes and heat transport equations, and therefore a different approach must be used that starts from the governing differential equations (§2.1). These are first written in potential form, because the scalars involved then transform as representations of four-dimensional black-and-white space groups. The differential operators are also transformed by the symmetry operators and their behaviour must be taken into account. The transformation properties of each term in the differential equation depends on that of both the operator and of the scalar function or functions on which it operates. The symmetry groups before and after the transition occurs have many elements in common, most of which need not be considered. It is important to discover the most compact description of the difference between the two groups involved, which is a factor group of the symmetry group with the greater number of symmetry elements. The nature of the transition is then controlled by the interrelationship between the irreducible representations of this factor group. The only practical difficulty is in finding the factor group of lowest possible order.

The application of group theory to the study of convective transitions involves the use of symbols and terminology that obscure the simplicity of the ideas involved. The analysis below depends on the interaction between the symmetry groups of the differential operators, which contain point symmetry elements and all translations, with those of the variables, which contain symmetry elements corresponding to discrete translations only, as well as point-group elements. Perturbation theory is used to analyse the nature of the transitions by expanding all quantities in terms of the amplitude ϵ of the perturbation. The Rayleigh number R is written as $R_0+\epsilon R_1+\epsilon^2 R_2\ldots$, where R_0 is the critical Rayleigh number for the transition, and terms of $O(\epsilon^3)$ and higher are ignored. The governing equations are then obtained, and in many cases contain a single term $R_1 f(x, y, z, t)$ that transforms differently from all the other terms. The value of this term must therefore be zero. In general, since $f \neq 0$, R_1 must be zero. The relationship between R and ϵ then requires

$$\epsilon = \pm((R-R_0)/R_2)^{\frac{1}{2}}$$

and the bifurcation is a pitchfork bifurcation if $R_2 \neq 0$. It is easy to demonstrate that the solvability condition is then also satisfied. The purpose of much of the analysis below is to consider the transformation properties of f and of the other functions without specifying the form of the functions themselves. By so doing the question of whether the equations involved are linear or nonlinear becomes irrelevant. But this approach requires the terminology and apparatus of group theory. Though all the applications discussed below are to convective circulations in plane layers, the same methods can be applied to any problems that can be described by differential equations, and are no harder to use on nonlinear partial differential equations than on linear ordinary differential equations.

The next section, 2, is concerned with the results from fluid dynamics, group theory and crystallography that are required. The crystallography (§2.3) is discussed in most detail because it is likely to be the least familiar to fluid dynamicists. Section 3 is concerned with the planform transitions observed by White (1981, 1988) in a convecting fluid whose viscosity is a strong function of temperature. Though his experiments were not designed to measure the Rayleigh numbers at which planform changes occurred, he observed a great variety of transitions from his initial circulations, and the nature of these bifurcations can easily be discovered using group theory. Most of the transitions discussed occur at Rayleigh numbers considerably above the critical, when the flow is thoroughly nonlinear. These transitions are easier to describe using group theory than is the bifurcation that occurs when convection starts. This section therefore illustrates the contrast between the use of factor groups and the usual perturbation theory. Section 4 is concerned with transitions between three-dimensional space groups, and is restricted to transitions between steady flows. Transitions discussed include the transition from the stationary state to rolls or hexagons, and how this transition is affected by boundary conditions and fluid properties. Other three-dimensional transitions are also analysed. Section 5 is concerned with transitions to oscillatory flows through Hopf bifurcations. Several examples are considered, all of which have the same factor group D_2. The manner in which pitchfork bifurcations could lead to aperiodic motions in space and time is outlined in §6.

2. Background

2.1. *Governing equations*

The transitions discussed in §§3–5 occur in convecting fluids driven by heat supplied from below. In the infinite-Prandtl-number experiments in §§3 and 4 the viscosity may be a function of temperature. Some of the transitions to time dependence in §5 involve fluids with finite Prandtl number and also double-diffusive effects. The governing equations are well known, and are simplest to use when they are scaled and when the velocity is expressed in terms scalar potentials S and ψ. Since the velocity $\boldsymbol{u}$ is solenoidal, it can be written as

$$\boldsymbol{u} = \boldsymbol{\nabla} \wedge \hat{\boldsymbol{z}}\psi + \boldsymbol{\nabla} \wedge \boldsymbol{\nabla} \wedge \hat{\boldsymbol{z}}S, \tag{2.1}$$

where $\hat{\boldsymbol{z}}$ is the unit vector pointing upwards, or

$$\boldsymbol{u} = (\partial_y \psi + \partial_x \partial_z S, \; -\partial_x \psi + \partial_y \partial_z S, \; -\nabla^2_{\mathrm{H}} S), \tag{2.2}$$

where $$\nabla^2_{\mathrm{H}} \equiv \partial^2_x + \partial^2_y.$$

When $\boldsymbol{u}$ is written in this way the flow is incompressible and mass is conserved. Conservation of heat requires

$$\partial_t T + \boldsymbol{u} \cdot \boldsymbol{\nabla} T = \nabla^2 T, \tag{2.3}$$

which leads to

$$\partial_t T + \boldsymbol{\nabla}_{\mathrm{H}} \partial_z S \cdot \boldsymbol{\nabla}_{\mathrm{H}} T - \nabla^2_{\mathrm{H}} S \partial_z T + \hat{\boldsymbol{z}} \cdot (\boldsymbol{\nabla} T \wedge \boldsymbol{\nabla} \psi) = \nabla^2 T, \tag{2.4}$$

where T is the temperature measured with respect to some reference temperature (often taken to be the mean temperature of the layer) and scaled using the temperature difference ΔT^* across the layer, and

$$\boldsymbol{\nabla}_{\mathrm{H}} = (\partial_x, \partial_y, 0).$$

The fourth term on the left of (2.4) may also be written as

$$\partial_y \psi \partial_x T - \partial_x \psi \partial_y T. \tag{2.5}$$

All lengths in these equations have been scaled by the thickness d of the convecting layer, and the time by the diffusive time d^2/κ_T, where κ_T is the thermal diffusivity. The corresponding equation for the conservation of a solute (such as salt) is

$$\partial_t s + \boldsymbol{u}\cdot\nabla s = \tau\nabla^2 s, \tag{2.6}$$

leading to

$$\partial_t s + \nabla_{\mathrm{H}}\partial_z S\cdot\nabla_{\mathrm{H}} s - \nabla^2_{\mathrm{H}} S\partial_z s + \hat{\boldsymbol{z}}\cdot(\nabla s\wedge\nabla\psi) = \tau\nabla^2 s, \tag{2.7}$$

where s is the solute concentration measured with respect to some reference concentration and scaled to the concentration difference Δs^* across the layer, and

$$\tau = \kappa_s/\kappa_T \tag{2.8}$$

is the reciprocal of the Lewis number, and κ_s is the diffusivity of the solute.

The equations corresponding to the conservation of momentum are more complicated. When the viscosity is constant and the density depends on both temperature and the concentration of the solute, the dimensionless form of the momentum equations is

$$\frac{1}{\sigma}(\partial_t \boldsymbol{u} + \boldsymbol{\omega}\wedge\boldsymbol{u}) = \nabla^2\boldsymbol{u} + (Ra T - R_s s)\hat{\boldsymbol{z}} - \nabla P, \tag{2.9}$$

where $\boldsymbol{\omega}$ is the vorticity

$$\begin{aligned}\boldsymbol{\omega} &= \nabla\wedge\boldsymbol{u}\\ &= (\partial_x\partial_z\psi - \nabla^2\partial_y S, \partial_y\partial_z\psi + \nabla^2\partial_x S, -\nabla^2_{\mathrm{H}}\psi);\end{aligned} \tag{2.10}$$

and

$$P = P^* + \tfrac{1}{2}u^2$$

where P^* is the pressure; Ra is the thermal Rayleigh number

$$Ra = \frac{g\alpha d^3\Delta T^*}{\kappa_T \nu}; \tag{2.11}$$

R_s is the solutal Rayleigh number

$$R_s = \frac{g\beta d^3\Delta s^*}{\kappa_s \nu}; \tag{2.12}$$

σ is the Prandtl number

$$\sigma = \nu/\kappa_T. \tag{2.13}$$

The density ρ is given by

$$\rho = \rho_0(1 - \alpha\Delta T^* T + \beta\Delta s^* s), \tag{2.14}$$

where ρ_0 is a constant. Equation (2.14) defines α and β, g is the acceleration due to gravity, ν the fluid viscosity, and ΔT^* and Δs^* the temperature and concentration differences between the lower (hot, large concentration) and the upper boundaries respectively. Evaluation of (2.9) gives

$$\begin{aligned}&\frac{1}{\sigma}(\partial_t(\nabla\wedge\hat{\boldsymbol{z}}\psi + \nabla\wedge\nabla\wedge\hat{\boldsymbol{z}}S) + (\nabla\wedge\nabla\wedge\hat{\boldsymbol{z}}\psi + \nabla\wedge\nabla\wedge\nabla\wedge\hat{\boldsymbol{z}}S)\wedge(\nabla\wedge\hat{\boldsymbol{z}}\psi + \nabla\wedge\nabla\wedge\hat{\boldsymbol{z}}S))\\ &\qquad = \nabla^2(\nabla\wedge\hat{\boldsymbol{z}}\psi + \nabla\wedge\nabla\wedge\hat{\boldsymbol{z}}S) + (Ra T - R_s s)\hat{\boldsymbol{z}} - \nabla P\end{aligned} \tag{2.15}$$

or

$$\frac{1}{\sigma}\{\hat{\boldsymbol{x}}[\partial_t(\partial_y \psi+\partial_x \partial_z S)-(\partial_y \partial_z \psi+\nabla^2\partial_x S)\nabla^2_{\mathrm{H}} S+(-\partial_x \psi+\partial_y \partial_z S)\nabla^2_{\mathrm{H}} \psi]$$
$$+\hat{\boldsymbol{y}}[\partial_t(-\partial_x \psi+\partial_y \partial_z S)+(\partial_x \partial_z \psi-\nabla^2\partial_y S)\nabla^2_{\mathrm{H}} S-(\partial_y \psi+\partial_x \partial_z S)\nabla^2_{\mathrm{H}} \psi]$$
$$+\hat{\boldsymbol{z}}[\partial_t(-\nabla^2_{\mathrm{H}} S)+(\partial_x \partial_z \psi-\nabla^2\partial_y S)(-\partial_x \psi+\partial_y \partial_z S)$$
$$-(\partial_y \partial_z \psi+\nabla^2\partial_x S)(\partial_y \psi+\partial_x \partial_z S)]\}$$
$$=\hat{\boldsymbol{x}}[\nabla^2(\partial_y \psi+\partial_x \partial_z S)-\partial_x P]+\hat{\boldsymbol{y}}(\nabla^2(-\partial_x \psi+\partial_y \partial_z S)-\partial_y P]$$
$$+\hat{\boldsymbol{z}}(-\nabla^2\nabla^2_{\mathrm{H}} S+RaT-R_s s-\partial_z P). \quad (2.16)$$

The curl of (2.9) gives

$$\frac{1}{\sigma}(\partial_t \boldsymbol{\omega}+\nabla\wedge(\boldsymbol{\omega}\wedge\boldsymbol{u})) = \nabla^2\boldsymbol{\omega}+\hat{\boldsymbol{z}}\wedge(Ra\nabla T-R_s\nabla s), \quad (2.17)$$

whose vertical component is

$$\frac{1}{\sigma}[-\partial_t\nabla^2_{\mathrm{H}}\psi+\partial_x(-\partial_y\psi\nabla^2_{\mathrm{H}}\psi-\nabla^2_{\mathrm{H}}\psi\partial_x\partial_z S+\partial_x\partial_z\psi\nabla^2_{\mathrm{H}}S-\nabla^2\partial_y S\nabla^2_{\mathrm{H}}S)$$
$$-\partial_y(-\partial_x\psi\nabla^2_{\mathrm{H}}\psi+\nabla^2_{\mathrm{H}}\psi\partial_y\partial_z S-\partial_y\partial_z\psi\nabla^2_{\mathrm{H}}S-\nabla^2\partial_x S\nabla^2_{\mathrm{H}}S)] = -\nabla^2\nabla^2_{\mathrm{H}}\psi. \quad (2.18)$$

The horizontal vorticity is maintained by the horizontal derivatives of T and s, but, as is well known, the vertical vorticity is only non-zero if the nonlinear terms involving S alone in (2.18) are non-zero. These reduce to

$$\partial_z^2\partial_x S\nabla^2_{\mathrm{H}}\partial_y S-\partial_z^2\partial_y S\nabla^2_{\mathrm{H}}\partial_x S. \quad (2.19)$$

A necessary but not sufficient condition for this term to be non-zero is that $S = S(x,y,z)$. Even then vertical vorticity is only generated if the Prandtl number is finite. The term involving R_s on the right of (2.16) is only present when the convection is doubly diffusive. The momentum equation simplifies greatly when the Prandtl number is infinite, since the left-hand side of (2.16) is then zero.

Several of the examples of time-dependent behaviour in §5 are concerned with two-dimensional flows. These are commonly described using a stream function $\Psi(x,z)$ where

$$\boldsymbol{u} = (\partial_z\Psi, 0, -\partial_x\Psi). \quad (2.20)$$

Equation (2.20) is identical to (2.2) if

$$\psi = 0, \quad \partial_x S = \Psi. \quad (2.21)$$

Therefore S and Ψ transform differently. It is convenient to use S and ψ in all discussions, but the corresponding behaviour of Ψ can be obtained from (2.21) if desired.

Some important transitions occur through Hopf bifurcations and concern the change from steady-state to time-dependent flow. Transitions observed experimentally can be discussed using the full convective equations (2.4), (2.7) and (2.16). But a well-known example of a Hopf bifurcation occurs in double-diffusive convection at the onset of convection. The linearized form of the equations governing doubly diffusive convection can be obtained from Veronis's (1965) expressions

$$d_t a_1 = -\sigma\pi^2(\alpha^2+1)\,a_1 + \frac{\sigma\alpha}{\pi(\alpha^2+1)}(Raa_3 - R_s a_5), \tag{2.22}$$

$$d_t a_3 = -\pi^2(\alpha^2+1)\,a_3 - \pi\alpha a_1, \tag{2.23}$$

$$d_t a_5 = -\tau\pi^2(\alpha^2+1)\,a_5 - \pi\alpha a_1, \tag{2.24}$$

where

$$S = -\frac{a_1}{\pi\alpha}\cos\pi\alpha x\,\sin\pi z, \tag{2.25}$$

$$\psi = 0, \tag{2.26}$$

$$T = a_3\cos\pi\alpha x\,\sin\pi z, \tag{2.27}$$

$$s = a_5\cos\pi\alpha x\,\sin\pi z. \tag{2.28}$$

The system that shows a greater variety of convective planforms than any other that has yet been studied is thermal convection in a fluid whose viscosity is a strong function of temperature (White 1981, 1988). Section 3 is principally concerned with a discussion of his results. The conditions he used produced a variation in the viscosity between the top and bottom surfaces, whose temperatures were fixed, of up to a factor of a thousand. Therefore the variation of viscosity cannot be treated as a perturbation, and the full equations must be used. The equations are simpler if the curl of the momentum equations is not taken. The equation governing the conservation of the two components of horizontal momentum then is

$$\begin{aligned} &2(\nabla_{\mathrm{H}}\eta\cdot\nabla_{\mathrm{H}})\,\nabla_{\mathrm{H}}\partial_z S + 2\eta\nabla_{\mathrm{H}}^2\nabla_{\mathrm{H}}\partial_z S + 2\partial_z\eta\nabla_{\mathrm{H}}(\partial_z^2 S - \nabla_{\mathrm{H}}^2 S) \\ &+\eta\nabla_{\mathrm{H}}(\partial_z^3 S - \nabla_{\mathrm{H}}^2\partial_z S) + 2(\hat{x}\partial_x - \hat{y}\partial_y)(\eta\partial_x\partial_y\psi) + (\hat{x}\partial_y + \hat{y}\partial_x)[\eta(\partial_y^2\psi - \partial_x^2\psi)] \\ &\qquad + \partial_z(\eta(\hat{x}\partial_y - \hat{y}\partial_x)\,\partial_z\psi) = -\nabla_{\mathrm{H}}P \end{aligned} \tag{2.29}$$

and that of vertical momentum is

$$\nabla_{\mathrm{H}}\eta\cdot\nabla_{\mathrm{H}}(\partial_z^2 S - \nabla_{\mathrm{H}}^2 S) + \eta\nabla_{\mathrm{H}}^2(\partial_z^2 S - \nabla_{\mathrm{H}}^2 S) + \hat{z}\cdot(\nabla_{\mathrm{H}}\eta\wedge\nabla_{\mathrm{H}}\partial_z\psi) = -RaT, \tag{2.30}$$

where the viscosity η^* is

$$\eta^* = \eta_0\,\eta(T) \tag{2.31}$$

and η_0 is the viscosity at the average of the top and bottom temperatures. The third term on the left of (2.30) may also be written as

$$(\partial_x\eta\partial_y - \partial_y\eta\partial_x)\,\partial_z\psi. \tag{2.32}$$

2.2. *Group theory*

The relevant results from group theory required below are proved in most introductory books on the subject, such as those by Ledermann (1973) and Hill (1975). Wigner's (1959) discussion is more directed towards physical applications, and that by Grossman & Magnus (1964) towards mathematical problems. Cotton (1963) provides a number of helpful examples of how abstract group theory may be used in chemistry.

A group G is defined by the multiplication table of its elements g_i, each of which must have an inverse g_i^{-1} that is also an element of the group. One element must be the identity element E, where $Eg_i = g_i$. The order of the group is the number of elements it contains. If two elements g_i and g_j are related by $g_i = g_k^{-1} g_j g_k$, where g_k is any element of the group, they belong to the same class, and g_i and g_j are called conjugate elements. The order of each class must be an integral factor of the order of the group. A compact description of a group is given in terms of generators, whose repeated application generates all the elements of the group. Different choices of generators are possible. A group contains a subgroup H with elements h_i if all products $h_i h_j$ are elements of both H and G. Conventionally the order of these operations is from left to right, which leads to the opposite convention for matrix multiplication being used in group theory to that of linear algebra (see Hill 1975, p. 48). The usual convention of linear algebra for matrices will be used below, since this is more familiar to fluid dynamicists. A subgroup is described as an invariant (or a normal or an isotropy) subgroup if $g_j^{-1} h_i g_j = h_k$ for every element of G, and is then written $H \lhd G$. To determine whether $H \lhd G$ it is only necessary to discover whether those elements of G that are not also elements of H satisfy this condition. If there is only one such element g_i and $g_i^2 = E$, Ledermann (1973, p. 62) shows that H must then always be an invariant subgroup. If H and K are two invariant subgroups of G, with only the identity element in common, and $g_i = h_j k_l$, G is called the direct product of H and K, and is written $G = H \otimes K$. If only K is an invariant subgroup then $G = HK$ is called the semidirect product of H and K, and is written $G = H \boxtimes K$. Many space groups are semidirect products of their subgroups, and it is therefore important to test whether either or both of H and K are invariant subgroups. It is also important to realize that $A \lhd H \lhd G$ does not require $A \lhd G$. The right coset of H in G generated by an element g_i of G is Hg_i, where g_i may be an element in common to both H and G. A very useful result is that cosets are either identical or have no elements in common. The left coset is $g_i H$ and is the same as the right coset if $H \lhd G$. An important result that is used repeatedly in the discussion below is that of a homomorphism between two groups G and H. G is homomorphic onto H if one and only one element of H corresponds to every element of G and at least one element of G corresponds to each element of H, and if the products of the corresponding elements in the two groups are the same. The two groups are isomorphic if only one element of G corresponds to each element of H. If $A \lhd G$, and the identity element of H corresponds to all the elements of the group A in G, then H is isomorphic to a factor group of G. This is written $G/A \cong H$. A is a maximal invariant subgroup of G if no group H exists (other than A or G) such that $A \lhd H \lhd G$. Correspondingly G is then a minimal supergroup of A. A has this property if and only if the factor group G/A has no subgroups. There is not necessarily a unique maximal invariant subgroup of G.

Applications of group theory are often simplified by making use of irreducible representations of groups, which may be one-dimensional real or complex numbers, or of higher dimensions, when the irreducible representations are matrices. Traces of these matrices are called characters. Representations are called faithful if each element of the group has a different representation. The irreducible representations of a number of finite abstract groups that occur as subgroups in the (infinite) space groups are given by Bradley & Cracknell (1972, table 5.1, p. 226), who also give the irreducible representations of all 230 space groups (table 5.7, p. 293). Once the irreducible representations have been determined they can be used to obtain projection operators. Any function can then be projected onto the irreducible representations to determine how it transforms. These operators can also be combined with the Fourier expansions in IT (1952) to obtain a complete representation of any irreducible representation of any factor group of a space group in terms of orthogonal functions. This procedure is illustrated in the Appendix by determining the functions that transform in the same way as do the irreducible representations of D_5 in the transition illustrated in figure 10.

One reason why the irreducible representations of groups are used so extensively in physics and chemistry is because of certain simple properties of their characters. In many problems the product of several terms which transform in different ways is needed. In the equations above, such products occur in the nonlinear terms, and also in the linear terms, which contain products between the linear differential operators and the variables. The great advantage of using irreducible representations is that the characters of the product can be obtained directly from the product of the characters of each of the irreducible representations of each term of the product (see Cotton 1963, p. 83). Since the product of two irreducible representations need not itself be irreducible, it is in general necessary to use the orthogonality of the characters of the irreducible representations to determine which are present in the product. The reason why this result is so important is that, since the characters are numbers, they are abelian. Hence the result is independent of the *order* in which the multiplication is carried out. Therefore from the point of view of the symmetry, the order of the terms in a differential equation is immaterial. It is also immaterial whether the irreducible representations in the product correspond to those of variables or of differential operators. From the point of view of symmetry there is therefore no difference between linear and nonlinear differential equations. These results are exploited in §2.4 to rewrite the fluid-dynamical equations in a form that emphasizes their symmetry.

However, before doing so it is necessary to discuss the problem of notation. Fortunately that for ordinary line, plane and space groups has been standardized by international agreement (IT 1983), and these international, or Hermann–Mauguin, symbols are used below. Similar tables have been produced for the four-dimensional space groups (Brown *et al.* 1978), but these are less easy to use. They include the space-group generators, but do not list the possible transitions. Brown *et al.*'s (1978) symbols are used for the four-dimensional space groups involved in time-dependant convective flows. The Schoenflies symbols are used for the factor groups, with Z_2 being the cyclic group of order 2, $D_2(\equiv V$ and $V_4) = Z_2 \otimes Z_2$ being Klein's Vierergruppe of order 4, D_3 the dihedral group of order 6, and D_5 the dihedral group of order 10. These symbols will be used for any groups that are isomorphic to these abstract groups, regardless of what physical operation the elements of the group represent. The notation used for the irreducible representations is simply to number them in the same way as do Bradley & Cracknell (1972). Use of the standard notation (see for instance Cotton 1963) for such representations would require different symbols for isomorphic groups, according to the nature of the group elements. Such a procedure would then obscure the simple relationships that become apparent when the same notation is used for all isomorphic groups.

2.3. *Line, plane and space groups*

Line, plane and space groups are the symmetry groups of periodic structures filling one-, two- or three-dimensional space respectively, and are listed with their generators in IT (1983). Each has a unit cell that fills space when repeated by translations. The conventional unit cell used in crystallography is not necessarily the smallest possible, and for many groups the conventional choice of axes (IT 1983) is not an orthogonal system. However all unit cells and generators listed below use orthogonal axes, defined in the appropriate figures. The generators are of two types:

displacements in the x-, y- and z-directions, with the length of the conventional unit cell taken to be a, b and unity in directions x, y and z, which generate a lattice called a Bravais lattice: and point operations. Symmetry elements that combine point operations and translations may also occur. The point operations consist of proper rotations C_2, C_3, C_4 and C_6, where C_n represents a rotation through $360/n°$, reflections m, and inversion $\bar{1}$. Values of n other than 2, 3, 4 and 6 cannot occur in periodic lattices if pseudocrystals are not considered (Landau & Lifshitz 1980, p. 405). Subscripts x, y and z are used to denote the rotation axis of C or the normal to the reflection plane. In crystallography the directions x, y and z are written [100], [010] and [001] respectively.

In two and three dimensions generators may also consist of combinations of reflections and displacements, known as glide planes g, and denoted a, b and c according to whether the displacement is parallel to the x-, y- or z-axis. When a possible choice of generators consists of a number of point-group operations carried out at the same point, together with a number of displacements, the space group is known as a symmorphic group. Some elements of the group may correspond to movement on glide planes, but if the group is symmorphic these operations can be written in terms of point symmetry elements and displacement elements. Non-symmorphic space groups cannot be generated from point-group elements operating at one point and translational elements, and contain generators which combine reflections and translations, corresponding to glide planes, or rotations and translations, corresponding to screw axes, or both.

When the symmetry of a space group changes it can do so in one of three ways (see Wondratschek in IT 1983, p. 727). A point-group symmetry element can be lost, without changing the translation group. Subgroups of this type are known as translationengleiche or t subgroups, and are written

$$\xrightarrow{t}$$

with a subscript denoting the point-group symmetry element that is lost. There are a finite number of such transitions. Another type of transition involves the loss of a translational symmetry element with the retention of the point-group symmetries. It is written

$$\xrightarrow{k}$$

and the resulting subgroup is known as a klassengleiche or k subgroup. Infinitely many k transitions can occur in any line, plane or space group. Notice that the symbol t or k refers to the symmetry elements that are *retained* in the transition. The most general transition involves the loss of both point and translational symmetry elements. Fortunately a theorem due to Hermann (1929) requires the maximal subgroup to be either a k or a t subgroup. IT (1983) list all the maximal t(I) and k(II) subgroups, and subdivide the k subgroups in various ways. But unfortunately no indication is given as to whether or not the subgroup is invariant. Each must therefore be checked. Figure 1 shows the invariant and ordinary subgroups of the two-dimensional point groups, and is helpful as a guide for transitions to t subgroups.

The short Hermann–Mauguin symbols are used to denote line, plane and space groups, and are discussed in IT (1983) table 2.4.1. The symbols start with a capital letter for space groups, a lower-case letter for plane groups and a script letter for line groups. The letter indicates the Bravais lattice of the group, with P (p and $\mathscr{p}$)

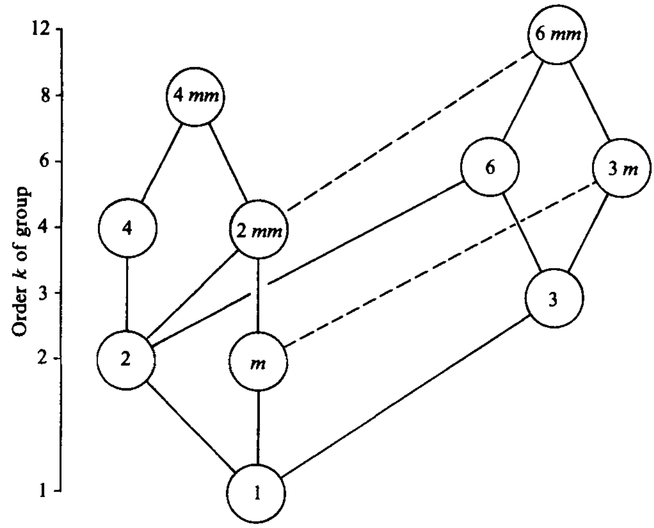

FIGURE 1. Maximal subgroups and minimal supergroups of the two-dimensional crystallographic point groups. The orders of the groups are given on the left. Where a group of higher order is joined to one of lower order by a solid line the lower-order group is an invariant subgroup of the higher. Where the line is dashed the subgroup is an ordinary subgroup (taken from IT 1983, figure 10.3.1 with modifications).

denoting a primitive lattice, A, B and C (c) face centred in the yz, xz and xy planes, and I and F (space groups only) body centred and face centred in all three planes. The next symbol denotes the rotational symmetry element normal to the plane (plane groups), or, in the case of space groups, a rotational element about a particular axis or mirror plane normal to this axis. If a space group has both, it is written as $P2/m$, for instance, to show the presence of both C_{2x} and m_x. The international symbols for these elements are 2 and m. Which axis it is whose symmetry follows the initial letter of the space-group symbol varies according to the lattice. For orthorhombic lattices the order is x, y, z, but for tetragonal and hexagonal lattices the C_4 and C_6 axes are taken to be in the z-direction and the space groups are written as $P4$ or $P6$. The symmetry direction of the last two symbols can be determined by reference to table 2.4.1, IT (1983, p. 15). The absence of any symmetry in one of these directions is denoted by a 1 in the full symbol, but is omitted in the short symbol. The space-group symbols for non-symmorphic groups can be recognized by the presence of numbers as subscripts denoting screw axes, and of the letters a, b, c, d or n, all of which refer to glide planes. In plane groups only glide lines occur and are denoted by g.

There are several advantages in using the international space-group symbols. They uniquely define the 230 space groups and 17 plane groups. Once the space group has been determined, the International Tables provide a list of generators, symmetry positions, k and t subgroups, and supergroups. They also provide the order of the factor group when the transition is to an invariant subgroup. Furthermore the earlier International Tables (IT 1952) contain the representation of each group in terms of Fourier series in two and three dimensions. These representations will considerably simplify the use of perturbation theory. Though the use of these tables is as yet unfamiliar to fluid dynamicists, they contain a variety of useful results.

For fluid dynamical use the tables must be extended in two ways, to take account of sign changes and of time-dependent behaviour. A sign change may be associated with any symmetry element, but does not occur in conventional crystallography because the electron density is always positive. It is, however, important if electron spin is of interest, as it is in magnetic materials. In fluid dynamics the temperature and velocity potentials can change sign. The resulting space groups are called black and white space groups and are discussed in a readable book by Shubnikov & Koptsik (1974). There are four types of such groups, three of which contain the sign reversal or antisymmetric operator $\mathbb{R}$. The first type, I, does not include $\mathbb{R}$, and consists of all the ordinary space groups in IT (1983). Groups belonging to the second type, II, contain $\mathbb{R}$ alone as a symmetry operator and are known as grey groups. Groups of types III and IV contain elements such as $\mathbb{R}m$ or $\mathbb{R}t$, denoted m' and t', which change the sign under reflection or translation, but do not contain $\mathbb{R}$ itself as an operator. When the element or elements producing the sign change are elements of a point group the space group is defined to be of type III, whereas when they are translations the group is of type IV. The black and white groups discussed below are all of types III and IV, which are isomorphic to groups of type I (Shubnikov & Koptsik 1974, pp. 272–5), since one or more of the generators is associated with $\mathbb{R}$. It is straightforward to decide which these generators are by using IT (1983). Bradley & Cracknell (1972) list the black and white or Shubnikov space groups together with the black and white generators, denoted by a prime. Their notation for these groups will be used here, and is based on the international symbols. Unfortunately the generators they use are not always the same as IT (1983). Tables like IT (1983) do not exist for the 1651 black and white space groups. However, because types III and IV are isomorphic to those in IT (1983) it is easy to use these tables to determine their subgroups. The same is not true of the four-dimensional black and white space groups that are required for the discussion of the oscillations of three-dimensional convective flows. Tables like IT (1983) for four-dimensional groups would be useful now that three-dimensional time-dependent computations have become practicable.

The groups listed in IT (1983) are all periodic in one, two or three dimensions. Some of the convective circulations discussed below are invariant under any translation, a symmetry that is often lost when a transition occurs. For instance the onset of convection involves a change from a one- to a two-dimensional flow, and the cross-roll instability from two to three dimensions. Infinite groups that contain all translations in one direction as symmetry elements will be denoted by a subscript zero, with P_{011} and p_{01} referring to groups containing all translations in the x-direction. It is also sometimes necessary to indicate complete axial symmetry about one axis, using the same order for the symbols as is used for orthorhombic space groups and ∞ to represent C_{∞}. For instance $P_{000}mm\infty/m'$ is a space group that is invariant under all translations, under any rotation about the z-axis and any reflection in any vertical plane, but changes sign when reflected in a horizontal plane.

The conventional international symbols refer to the coordinate system defined in IT (1983). This is not necessarily the same as that used in the fluid-dynamical equations. No difficulty arises in the use of plane groups to describe planform changes, since the z-axis is taken to be vertical in both fluid dynamics and crystallography. In the case of space groups, however, the coordinate systems are often different. The space-group symbol will then be written using the standard fluid-dynamical coordinate system and is unconventional. The conventional symbol therefore follows in brackets, so that the relevant space group can be found in IT (1983). Since the symmetry positions and generators are also given in the crystallographic axes, these must also be converted (see §4). One further crystallographic convention that is used is to denote $-x$ as $\bar{x}$.

The first step in any discussion of a transition requires the space group or plane groups of both states. It is generally easy to decide which is the appropriate lattice, to identify the axis of greatest symmetry and any mirror planes. This information together with table 2.4.1 of IT (1983) is sufficient to eliminate most possibilities and to write out the first one or two entries of the international symbols. All appropriate space groups must then be examined to find one which has all and only the symmetry elements of the convective circulation. Glide planes not perpendicular to the axes are easy to miss. Then the convective pattern is assigned to a subgroup of the space group to which it properly belongs. When the appropriate groups have been found it is then necessary to determine those generators common to both groups, and the generator or generators possessed only by the space or plane group of higher symmetry. It is frequently necessary to use a different choice of generators from that given in IT (1983). These operations are discussed in detail in §3.

2.4. *Symmetry of variables and differential operators*

The results of the first three sections must now be combined to discover how the operators and variables in the differential equations behave when transformed by the elements of the space groups. The symmetry groups of the operators will be considered first. All space-group transformations correspond to operations that preserve lengths and angles, and therefore do not affect the form of the differential operators, though their signs may change. Differential operators therefore transform in the same way as do those irreducible representations of the Euclidean groups E_n, where n is the dimension of the space involved, that are invariant with respect to all translations. Space groups are subgroups of these Lie groups, and the variables transform in the same way as do the irreducible representations of these subgroups. To determine which representation of a space group is appropriate for each differential operator it is therefore only necessary to discover how an operator behaves under the point-group elements of the space groups, since all are invariant under translations. It is not necessary to find the irreducible representation of the full Lie group that transforms in the same way as does the operator, because this representation must also transform like the operator under the elements of any subgroup of the group.

The symmetry groups of the operators are conveniently shown using a simple representation of the Lie group E, $(\hat{\boldsymbol{x}} \wedge \hat{\boldsymbol{y}})$, $\hat{\boldsymbol{z}}$ and $\hat{\boldsymbol{t}}$ that is invariant under $C_{\infty z}$. $(\hat{\boldsymbol{x}} \wedge \hat{\boldsymbol{y}})$ changes sign under m_x or m_y, but not under m_z or m_t, whereas $\hat{\boldsymbol{z}}$ changes sign only under m_z, and $\hat{\boldsymbol{t}}$ only under m_t. These are the only point-group elements that occur in the space groups considered below. This representation is invariant under any translation and under $\mathbb{R}$, and can therefore transform as do the differential operators. But it is only their transformational behaviour, not their direction, that is represented.

The behaviour of the differential operators under reflection can be determined by inspection, but that under $C_{\infty z}$ is sometimes not obvious and must be determined by using rotation matrices. Rotation through $+\theta$ about the z-axis changes (x, y) to (x', y') where

$$x' = x\cos\theta - y\sin\theta, \quad y' = x\,sin\,\theta + y\cos\theta. \tag{2.33}$$

The common convention in group theory that is used here is that elements of point groups leave the axes fixed in space and rotate the vectors. The orthogonal matrix describing the rotation (2.33) is then the inverse of that for the rotation of the axes. Hence

$$\left.\begin{aligned} \hat{x}' &= \hat{x}\cos\theta - \hat{y}\sin\theta, \quad \hat{y}' = \hat{x}\sin\theta + \hat{y}\cos\theta, \\ \partial_{x'} &= \cos\theta\partial_x - \sin\theta\partial_y, \quad \partial_{y'} = \sin\theta\partial_x + \cos\theta\partial_y. \end{aligned}\right\} \tag{2.34}$$

It is then straightforward to show that ∂_z, $\mathbf{\nabla}_\mathrm{H}$, $\mathbf{\nabla}$, ∇^2_H, ∇^2 and $\mathbf{\nabla}\wedge$ are invariant under any rotation about a vertical axis. Hence the linear operators in all the symmetry equations are invariant under $C_{\infty z}$. The nonlinear terms in (2.4), (2.7) and (2.15) are also invariant under $C_{\infty z}$. This result was also proved by Sattinger (1978) using (2.3) and (2.9), but is more obvious when the equations are written in terms of potentials. It is also to be expected that the last three terms on the left-hand side of (2.29) will be invariant under $C_{\infty z}$, since no particular horizontal axes is defined by the dynamics, but this result is less obvious from the form of the operators. In a rotated frame the x- and y-dependence of the last operator on the left of (2.29) is

$$\hat{x}'\partial_{y'} - \hat{y}'\partial_{x'}; \tag{2.35}$$

substitution of (2.34) gives

$$\hat{x}\partial_y - \hat{y}\partial_x \tag{2.36}$$

and therefore this operator also is invariant under $C_{\infty z}$. The fifth and sixth operators must be taken together and after some algebra may be shown also to be invariant under $C_{\infty z}$.

To discover the appropriate representation, the differential operators must first be expressed in terms of unit vectors and scalar operators. Their behaviour under m_x, m_y, m_z and m_t can then be determined. For linear terms this procedure is easily carried out:

$$\left.\begin{array}{rl} \partial_z & \hat{z} \\ \partial_t & \hat{t} \\ \mathbf{\nabla}_\mathrm{H} \equiv \hat{x}\partial_x + \hat{y}\partial_y & E \\ \mathbf{\nabla} \equiv \mathbf{\nabla}_\mathrm{H} + \hat{z}\partial_z & E \\ \nabla^2_\mathrm{H} \equiv \partial^2_x + \partial^2_y & E \\ \nabla^2 \equiv \nabla^2_\mathrm{H} + \partial^2_z & E \\ \mathbf{\nabla}\wedge\hat{z} \equiv \hat{x}\partial_y - \hat{y}\partial_x & (\hat{x}\wedge\hat{y}) \\ \mathbf{\nabla}\wedge\mathbf{\nabla}\wedge\hat{z} \equiv \hat{x}\partial_x\partial_z + \hat{y}\partial_y\partial_z - \hat{z}\nabla^2_\mathrm{H} & \hat{z} \\ \mathbf{\nabla}\wedge\mathbf{\nabla}\wedge\mathbf{\nabla}\wedge\hat{z} \equiv -\hat{x}\nabla^2\partial_y + \hat{y}\nabla^2\partial_x & (\hat{x}\wedge\hat{y}) \end{array}\right\} \tag{2.37}$$

where the transformational behaviour is given in the last column. There is no relationship between the direction of a vector operator and that of the representation that transforms in the same way. The nonlinear terms in (2.4) can be expressed as

$$\left.\begin{array}{rl} \mathbf{\nabla}_\mathrm{H}\partial_z S\cdot\mathbf{\nabla}_\mathrm{H} T - \nabla^2_\mathrm{H} S\partial_z T & \hat{z}\times S\times T \\ \partial_y\psi\partial_x T - \partial_x\psi\partial_y T & (\hat{x}\wedge\hat{y})\times\psi\times T, \end{array}\right\} \tag{2.38}$$

where the product sign $\times$ means that the transformational behaviour is that of the two terms on either side. Since (2.38) is only concerned with the transformational behaviour, which in turn depends only on the abelian products of the corresponding characters of the irreducible representations (see §2.2), the order in which the terms are written is unimportant. Equation (2.4) then becomes

$$\hat{t}\times T + \hat{z}\times S\times T + (\hat{x}\wedge\hat{y})\times\psi\times T = T. \tag{2.39}$$

Equations such as (2.39) will be referred to as symmetry equations. The signs are not meaningful and will always be shown positive. The dimensionless constants are retained where present, but, because they are constants and are unaffected by symmetry operations, they are not shown with product signs. For reasons explained below, it is convenient to construct the product of (2.39) with $\hat{z}$. The irreducible representations Γ_n that transform in the same way as do $\hat{z}$ and $(\hat{x} \wedge \hat{y})$ satisfy $\Gamma_n^2 = \Gamma_1$, where Γ_1 is the trivial representation, in all the examples considered below. Therefore $\hat{z} \times \hat{z} = E$ and (2.39) becomes

$$\hat{t} \times \hat{z} \times T + S \times T + (\hat{x} \wedge \hat{y}) \times \hat{z} \times \psi \times T = \hat{z} \times T. \tag{2.40}$$

Similarly (2.7) becomes

$$\hat{t} \times \hat{z} \times s + S \times s + (\hat{x} \wedge \hat{y}) \times \hat{z} \times \psi \times s = \tau \hat{z} \times s. \tag{2.41}$$

The momentum equation (2.16) becomes

$$\frac{1}{\sigma}(\hat{t} \times ((\hat{x} \wedge \hat{y}) \times \psi + \hat{z} \times S) + (\hat{x} \wedge \hat{y}) \times \hat{z} \times \psi \times S + S \times S + \psi \times \psi)$$
$$= \hat{z} \times S + (\hat{x} \wedge \hat{y}) \times \psi + Ra\hat{z} \times T + R_s \hat{z} \times s + P, \tag{2.42}$$

and that for the vertical vorticity (2.20) is

$$\frac{1}{\sigma}(\hat{t} \times \psi + (\hat{x} \wedge \hat{y}) \times (\psi \times \psi + S \times S) + \hat{z} \times \psi \times S) = \psi. \tag{2.43}$$

If (2.43) is multiplied by $(\hat{x} \wedge \hat{y})$ then, since $(\hat{x} \wedge \hat{y}) \times (\hat{x} \wedge \hat{y}) = E$,

$$\frac{1}{\sigma}(\hat{t} \times (\hat{x} \wedge \hat{y}) \times \psi + \psi \times \psi + S \times S + (\hat{x} \wedge \hat{y}) \times \hat{z} \times \psi \times S) = (\hat{x} \wedge \hat{y}) \times \psi. \tag{2.44}$$

Equation (2.44) is consistent with (2.42), as it must be, but is more restrictive because it only contains terms involving S and ψ.

When the viscosity is a function of temperature (2.29) and (2.30) give a symmetry equation that may be written

$$\hat{z} \times \eta \times S + (\hat{x} \wedge \hat{y}) \times \eta \times \psi = Ra\hat{z} \times T + P. \tag{2.45}$$

The behaviour of the viscosity is most easily obtained by expanding (2.31) in a Taylor series:

$$\eta = E + T + T \times T + T \times T \times T + \ldots. \tag{2.46}$$

The symmetry groups of the variables are controlled by the boundary conditions and by (2.2). If the right-hand side of (2.1) is to transform like a vector, ψ and S must behave in the appropriate way. Under m_x

$$x \rightarrow -x, \quad u_x \rightarrow -u_x, \quad u_y \rightarrow u_y, \; u_z \rightarrow u_z. \tag{2.47}$$

These conditions require $\quad \psi \rightarrow -\psi, \quad S \rightarrow S. \qquad (2.48)$

Since the operators are unchanged under $C_{\infty z}$, these conditions apply to any reflection in a vertical plane. Under m_z however

$$z \rightarrow -z, \quad u_x \rightarrow u_x, \quad u_y \rightarrow u_y, \quad u_z \rightarrow -u_z. \tag{2.49}$$

These conditions are satisfied if

$$\psi \rightarrow \psi, \quad S \rightarrow -S. \tag{2.50}$$

The symmetry equations (2.40)–(2.42) are also invariant under m_x if

$$T \rightarrow T, \quad s \rightarrow s, \tag{2.51}$$

and under m_z if

$$T \rightarrow -T, \quad s \rightarrow -s. \tag{2.52}$$

S, s and T therefore all have the same spatial symmetry, which is different from that of the two-dimensional stream function Ψ.

It is now clear why the symmetry equations have been written in a form that contains only $\hat{z} \times S$, $\hat{z} \times T$, $\hat{z} \times s$, $(\hat{x} \wedge \hat{y}) \times \psi$, $S \times S$, $S \times T$, $S \times s$ and $\psi \times \psi$. None of these symmetry expressions change sign on reflection in horizontal or vertical planes, and therefore none have black and white symmetry.

When the viscosity depends on temperature, (2.46) shows that, under m_x, $\eta \rightarrow \eta$ and the equations are invariant. However, under m_z, $T \rightarrow -T$ and

$$\eta \rightarrow E - T + T \times T - T \times T \times T + \ldots \tag{2.53}$$

and the equations are not invariant.

Since all components of the velocity reverse under m_t, and since the differential operators in (2.1) are unchanged, $S \rightarrow -S$ and $\psi \rightarrow -\psi$ under this operation. As is well known, and as is clear from (2.40)–(2.42), the differential operators in the equations are not invariant under m_t when the solutions are time dependent.

The last symmetry operation of concern is the sign-change operator $\mathbb{R}$. By definition

$$\mathbb{R}S \rightarrow -S, \quad \mathbb{R}s \rightarrow -s, \quad \mathbb{R}T \rightarrow -T, \quad \mathbb{R}\psi \rightarrow -\psi. \tag{2.54}$$

It is straightforward to show that these symmetry conditions are compatible with $\boldsymbol{u} = 0$ and $\boldsymbol{u}.\hat{z} = 0$ on $z = 0, 1$, and with $\sigma_{xz} = \sigma_{yz} = 0$ if ψ is independent of x and y on the boundary and has a symmetry element m_z, and if $\nabla^2 S = 0$ on both boundaries.

Perturbation theory can now be used to obtain the symmetry equations governing the behaviour of perturbations. If all variables may be expanded in terms of a small parameter ϵ, then

$$\left.\begin{aligned} S = S_0 + \epsilon S_1 + \ldots, \quad s &= s_0 + \epsilon s_1 + \ldots, \\ T = T_0 + \epsilon T_1 + \ldots, \quad \psi &= \psi_0 + \epsilon \psi_1 + \ldots, \\ Ra &= Ra_0 + \epsilon Ra_1 + \epsilon^2 Ra_2 + \ldots, \\ R_s &= R_{s0} + \epsilon R_{s1} + \epsilon^2 R_{s2} + \ldots, \\ P &= P_0 + \epsilon P_1 + \ldots. \end{aligned}\right\} \tag{2.55}$$

In crystallography ϵ is known as the order parameter. Taking the zeroth-order solution to be a steady state that is being perturbed, (2.40) gives

$O(\epsilon^0)$

$$S_0 \times T_0 + (\hat{x} \wedge \hat{y}) \times \hat{z} \times \psi_0 \times T_0 = \hat{z} \times T_0; \tag{2.56}$$

$O(\epsilon^1)$

$$\hat{t} \times \hat{z} \times T_1 + (S_0 \times T_1 + T_0 \times S_1) + (\hat{x} \wedge \hat{y}) \times \hat{z} \times (\psi_0 \times T_1 + T_0 \times \psi_1) = \hat{z} \times T_1. \tag{2.57}$$

Equation (2.41) becomes

$O(\epsilon^0)$

$$S_0 \times s_0 + (\hat{\boldsymbol{x}} \wedge \hat{\boldsymbol{y}}) \times \hat{\boldsymbol{z}} \times \psi_0 \times s_0 = \tau \hat{\boldsymbol{z}} \times s_0; \tag{2.58}$$

$O(\epsilon^1)$

$$\hat{\boldsymbol{t}} \times \hat{\boldsymbol{z}} \times s_1 + (S_0 \times s_1 + s_0 \times S_1) + (\hat{\boldsymbol{x}} \wedge \hat{\boldsymbol{y}}) \times \hat{\boldsymbol{z}} \times (\psi_0 \times s_1 + s_0 \times \psi_1) = \tau \hat{\boldsymbol{z}} \times s_1. \tag{2.59}$$

Equation (2.42) gives

$O(\epsilon^0)$

$$\frac{1}{\sigma}[(\hat{\boldsymbol{x}} \wedge \hat{\boldsymbol{y}}) \times \hat{\boldsymbol{z}} \times \psi_0 \times S_0 + (\psi_0 \times \psi_0 + S_0 \times S_0)]$$
$$= (\hat{\boldsymbol{x}} \wedge \hat{\boldsymbol{y}}) \times \psi_0 + \hat{\boldsymbol{z}} \times S_0 + Ra_0\, \hat{\boldsymbol{z}} \times T_0 + R_{s0}\, \hat{\boldsymbol{z}} \times s_0 + P_0 \tag{2.60}$$

$O(\epsilon^1)$

$$\frac{1}{\sigma}\{\hat{\boldsymbol{t}} \times [(\hat{\boldsymbol{x}} \wedge \hat{\boldsymbol{y}}) \times \psi_1 + \hat{\boldsymbol{z}} \times S_1]$$
$$+ (\hat{\boldsymbol{x}} \wedge \hat{\boldsymbol{y}}) \times \hat{\boldsymbol{z}} \times (S_0 \times \psi_1 + \psi_0 \times S_1) + (\psi_0 \times \psi_1 + S_0 \times S_1)\}$$
$$= (\hat{\boldsymbol{x}} \wedge \hat{\boldsymbol{y}}) \times \psi_1 + \hat{\boldsymbol{z}} \times S_1 + Ra_0\, \hat{\boldsymbol{z}} \times T_1 + Ra_1\, \hat{\boldsymbol{z}} \times T_0$$
$$+ R_{s0}\, \hat{\boldsymbol{z}} \times s_1 + R_{s1} \hat{\boldsymbol{z}} \times s_0 + P_1. \tag{2.61}$$

Equation (2.45) gives

$O(\epsilon^0)$

$$\hat{\boldsymbol{z}} \times \eta_0 \times S_0 + (\hat{\boldsymbol{x}} \wedge \hat{\boldsymbol{y}}) \times \eta_0 \times \psi_0 = Ra_0\, \hat{\boldsymbol{z}} \times T_0 + P_0; \tag{2.62}$$

$O(\epsilon^1)$

$$\hat{\boldsymbol{z}} \times \eta_0 \times S_1 + \hat{\boldsymbol{z}} \times S_0 \times \eta_1 + (\hat{\boldsymbol{x}} \wedge \hat{\boldsymbol{y}}) \times (\eta_0 \times \psi_1 + \psi_0 \times \eta_1)$$
$$= Ra_0\, \hat{\boldsymbol{z}} \times T_1 + Ra_1\, \hat{\boldsymbol{z}} \times T_0 + P_1. \tag{2.63}$$

Equation (2.46) becomes

$O(\epsilon^0)$

$$\eta_0 = E + T_0 + T_0 \times T_0 + \ldots; \tag{2.64}$$

$O(\epsilon^1)$

$$\eta_1 = (E + T_0 + T_0 \times T_0 + \ldots) \times T_1. \tag{2.65}$$

These symmetry equations are formal statements of the 'symmetry' that anyone who has worked with equations such as (2.16) will have noticed, and which provides a convenient check on correctness of algebraic operations. Several features of these equations are of interest. Perhaps the most important is that the three momentum equations have been reduced to one symmetry equation by the use of potentials. Another is the presence of P_0 and P_1 in (2.60)–(2.63). Since these terms could have been eliminated by taking the curl of the momentum equations, these must be satisfied when P_0 and P_1 are omitted. The equations do, however, determine the transformational behaviour of P_0 and P_1.

2.5. *Factor groups governing a transition*

It is now possible to define what is meant by a factor group of a convective transition. It is constructed by first finding the group G containing all the symmetry elements of all the operators and variables of the relevant perturbation equations to $O(\epsilon^0)$ and $O(\epsilon^1)$. There is always an invariant subgroup H of G that is generally most easily found from the symmetry group of the perturbation. The group governing the transition $\mathbb{T}$ is isomorphic to the factor group G/H. Its importance is that the behaviour of each operator and variable must correspond to that of one or more of the irreducible representations of $\mathbb{T}$. The behaviour of the combination of an operator and a variable, shown by a multiplication sign in the symmetry equation, can be found from that of the product of the representations involved if these are one-dimensional. If the irreducible representations are matrices, then the product signs represent the multiplication of the characters of these matrices. The order of the factor group can be reduced by considering the symmetry group of operators and variables taken together, and the symmetry equations (2.56)–(2.63) have been written to exploit this property. If the symmetry group of T_1 is (for instance) $Pmmm'$, then that of $\hat{z}\times T_1$ is $Pmmm$. The same result holds for ψ_1, whose symmetry group must be $Pm'm'm$ in the same system, whereas that of $(\hat{x}\wedge\hat{y})\times\psi_1$ is $Pmmm$. The factor group therefore need not contain m' as an element. Since the number of entries in the table of irreducible representations increases as approximately 2^n, where n is the number of generators, as a practical matter it is important to keep the order of $\mathbb{T}$ as small as possible.

3. Planform transitions when $\eta = \eta(T)$

3.1. *k transitions in the absence of* C_{3z}

The simplest of all the transitions that White (1981) studied he called the mosaic instability. The two planforms involved are shown in figure 2(*a* and *b*). Only the cold sinking sheets are shown as solid lines. He found that figure 2(*a*) was stable when $Ra = 14800$, $\alpha = 3.70$ and that figure 2(*b*) was stable when $Ra = 63350$, $\alpha = 3.74$, where α is the wavenumber, when the viscosity variation between the top and bottom boundaries was 50. The critical Rayleigh number for this instability is estimated to be about 20000 when $\alpha = 3.7$.

It is first necessary to decide to what plane groups the two planforms belong. Both patterns clearly have C_{4z} axes and two mirror planes, and therefore contain the point group $4mm$. The symmetry operations of $p4mm$, illustrated in figure 2(*c*), shows that four mirror planes should intersect on the C_{4z} axis. Inspection of figure 2(*a* and *b*) shows that they do. The glide lines and C_{2z} axes are also present. Therefore both patterns belong to the plane-group type $p4mm$, though their primitive unit cells, outlined by dashed lines, are of different size and orientation. Though $p4mm$ is a symmorphic group, it contains rotational symmetry elements that act at several points and also glide lines. These symmetry elements are a consequence of the periodicity of the lattice and the presence of a $4mm$ point group. IT (1983) gives as generators

$$1, \quad t_1 = (a,0), \quad t_2 = (0,a), \quad C_{4z}, \quad m_x. \tag{3.1}$$

m_x and C_{4z} act at the origin, which is taken to be on the C_{4z} axis, and the length of the sides of the primitive unit cell in figure 2(*b*) are taken to be a. Since both patterns have symmetry group $p4mm$ the transition is of type k. However the change in the

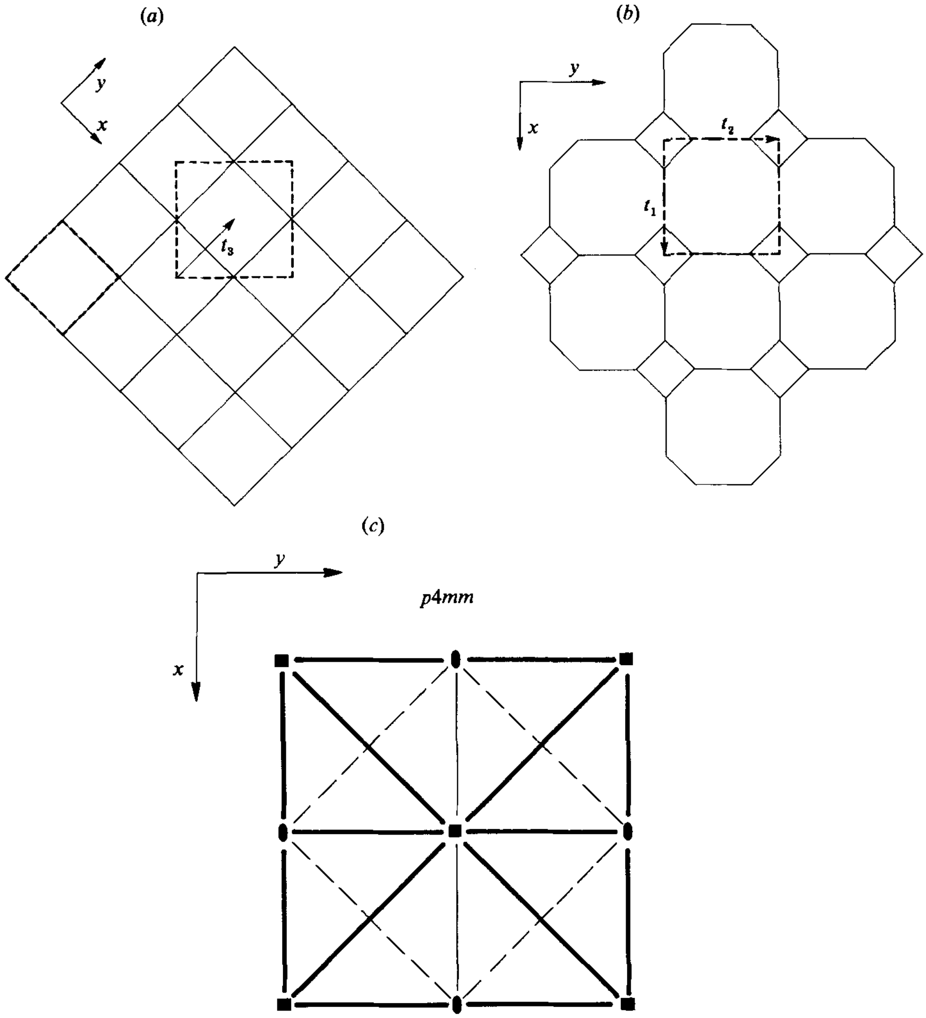

FIGURE 2. Planform transition by loss of t_3, a k-type transition called the mosaic instability by White (1988). The primitive unit cell and the larger unit cell, which is the primitive unit cell of (*b*), are shown dashed. The additional symmetry element present in (*a*) is the centring translation t_3. (*c*) Shows the symmetry elements of the primitive cell *p*4*mm*. Squares are tetrad, ellipses are diad axes. Heavy lines are mirror planes and dashed lines are glide planes.

symmetry group does not correspond to a doubling in the length of the side of the primitive cell in figure 2(*a*), since there would then be no change in orientation of the axes and the area of the cell would increase in size by a factor of four. The symmetry element that is lost is the translation labelled $t_3 = (a/2, a/2)$ in figure 2(*a*), which is present in figure 2(*a*) but absent in figure 2(*b*). Such elements that are present in one group but not in the other will be referred to as active elements. The larger unit cell marked by dashed lines in figure 2(*a*) will cover the plane when repeated, but is not a primitive cell. It is therefore written as *c*4*mm*, using *c* to indicate the presence of centring generator t_3. The next concern is whether *p*4*mm* of figure 2(*b*) is an invariant subgroup of *c*4*mm*. This question can be answered by discovering whether

$$t_3 g_i t_3^{-1} = g_j \tag{3.2}$$

is satisfied, where g_i and g_j are elements of $p4mm$. The first three generators clearly satisfy (3.2). That the second three do also is most easily seen by using the rotation matrices corresponding to C_{4z} and m_x:

$$C_{4z} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad m_x = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{3.3}$$

to compare the effect of transforming a vector (x, y) to that of one shifted to $(x+a/2, y+a/2)$ before rotation and then shifted back by $(-a/2, -a/2)$ after. The matrices (3.3) form a faithful representation of the point group. These operations show that

$$t_3 C_{4z} t_3^{-1} = C_{4z} t_1^{-1}, \quad t_3 m_x t_3^{-1} = m_x t_1^{-1}. \tag{3.4}$$

Clearly the right-hand sides of (3.4) are elements of $p4mm$, which is therefore an invariant subgroup. The symmetry group of figure 2(*a*) can then be written in terms of cosets:

$$\{c4mm\} = \{p4mm\} + t_3\{p4mm\}. \tag{3.5}$$

Though there are infinitely many elements t_3^n of the translational group with generator t_3, there are only two cosets. This is because $t_3^2 = t_1 t_2$. Therefore elements with n even are elements of the first coset, and those with n odd are elements of the second. Because $p4mm$ contains the identity element, possible cosets such as $t_3^n\{p4mm\}$ must contain the element t_3^n. But this element is an element of one of the two cosets in (3.5). Hence this possible coset must be identical to one of those in (3.5), and the factor group must be isomorphic to Z_2. The irreducible representations can be written as

Z_2	$E(p4mm)$	t_3	
Γ_1	1	1	$c4mm$
Γ_2	1	-1	$p4mm$

(3.6)

The notation used here shows the abstract group isomorphic to the factor group at the top left, with the cosets that form the representation of this group in line to the right. The invariant subgroup involved is written in brackets following E. The identity element E in the factor group is the entire symmetry group $p4mm$. The other element of the factor group is the product of all elements of this group with t_3. The irreducible representations are numbered Γ_1 to Γ_n, with Γ_1 always being the trivial representation. Tables like (3.6) show the characters, which in this case are the same as the irreducible representations because these are one-dimensional. When the representations are two-dimensional the representations of the generators will also be given. The last column on the right shows the plane group that transforms in the same way as does the irreducible representation on the same line. In this example the irreducible representation Γ_2 is a faithful representation of the factor group. It is clear that any element of the coset $t_3\{p4mm\}$ could be used as the active element. The procedure necessary to determine the factor group has been explained in detail in this case because it is fundamental to all the applications discussed below. In fact, since there are only two elements in the factor group, $p4mm$ must be an invariant subgroup. The relevant steps will in the future only be discussed when they differ from those in this example.

The symmetry equations in steady state for the circulations in figure 2 are

$$S_0 \times T_1 + T_0 \times S_1 + (\hat{x} \wedge \hat{y}) \times \hat{z} \times (\psi_0 \times T_1 + T_0 \times \psi_1) = \hat{z} \times T_1, \tag{3.7}$$

together with (2.56), (2.62)–(2.65). $(\hat{x} \wedge \hat{y})$ and $\hat{z}$ are invariant under t_3. All operators are therefore passive and transform as does Γ_1. The symmetry groups in figure 2(*a* and *b*) are those of T_0 and T_1 respectively. Since T_0 transforms as Γ_1, (2.64) shows that η_0 must also. This statement will be abbreviated to $\eta_0 = \Gamma_1$. Then (2.62) requires $S_0 = \Gamma_1$. The argument above shows that $T_1 = \Gamma_2$ under t_3. Since $\Gamma_1 \times \Gamma_2 = \Gamma_2$, (2.65) requires $\eta_1 = \Gamma_2$ and (3.7) requires $S_1 = \Gamma_2$. Turning to (2.63), $\eta_0 \times S_1$, $S_0 \times \eta_1$, $Ra_0 T_1 = \Gamma_2$ but $Ra_1 T_0 = \Gamma_1$. Because there are no other terms in (2.63) that transform as does Γ_1, $Ra_1 T_0 = 0$. Since $T_0 \neq 0$, Ra_1 must be zero. Provided $Ra_2 \neq 0$, to $O(\epsilon^2)$ the expression for Ra in (2.55) may be written as

$$\epsilon = \pm[(Ra - Ra_0)/Ra_2]^{\frac{1}{2}}, \tag{3.8}$$

where Ra_0 is the critical Rayleigh number for this transition. Because of the orthogonality of the characters of the different representations, the solvability condition is also satisfied if T_1 contains no component that transforms as does Γ_1. Equation (3.8) is the standard expression for the perturbation amplitude near a pitchfork bifurcation and corresponds to the steady-state solution of the Landau equation.

The Landau equation itself may be obtained by allowing $\epsilon = \epsilon(t)$, but in this and other examples transient behaviour will not be considered. This argument therefore shows that the transition in figure 2 is a pitchfork, rather than a transcritical or a Hopf, bifurcation, and does so without attempting to describe the functional form of the temperature distribution. It is only necessary to describe the change in symmetry. The essential part of the argument depends on the symmetry groups of the terms on the left-hand side of (2.63). Provided these all transform as does one of the irreducible representations of Z_2, and this representation is different from that of T_0, then Ra_1 must be zero and (3.8) holds. It is not, however, possible to determine the sign nor the magnitude of Ra_2 (or even whether it differs from 0), and therefore whether the pitchfork bifurcation is sub- or supercritical.

The above discussion ignored the terms in ψ and is therefore incomplete. All terms containing ψ are associated with $(\hat{x} \wedge \hat{y})$. Since both ψ and $(\hat{x} \wedge \hat{y})$ change sign under m_x, the product is invariant under m_x. Hence the symmetry equations are satisfied if

$$(\hat{x} \wedge \hat{y}) \times \psi_0 = \Gamma_1, \quad (\hat{x} \wedge \hat{y}) \times \psi_1 = \Gamma_2$$

and the condition on Ra_1 remains.

The irreducible representation Γ_2 corresponding to T_1 and S_1 changes sign under t_3. It is therefore invariant under t_3' and the space group of the perturbation is a type IV Shubnikov space group. In all pitchfork bifurcations with factor groups Z_2 the symmetry group of the perturbation has a black and white element that is not an element of the symmetry group of the unperturbed flow. *k* transitions result in type IV Shubnikov groups, *t* transitions in type III.

Another well known *k* transition observed by White (1988, figure 23) is from rolls to rectangular cells, shown in figure 3. Rolls have a symmetry group $p_{01}2mm$, because they are invariant to any translation in the x-direction. The symmetry group of the bimodal pattern is $p2mm$, since the wavenumber of the cross-rolls is in general different from that of the basic flow. Generators of $p2mm$ are E, $t_1 = (a, 0)$, $t_2 = (0, b)$, C_{2z}, m_y, where $a \neq b$. Taking $t_3 = (a/2, 0)$ yields two cosets for the same reason as before, and the invariant subgroup is $p2mm$. But the infinite group $p_{01}2mm$ is invariant under any translation in the x-direction. It is therefore unlike the space group $c4mm$ considered in the first example, which, though infinite, was only invariant under a discrete infinite set of translations. There are therefore infinitely many

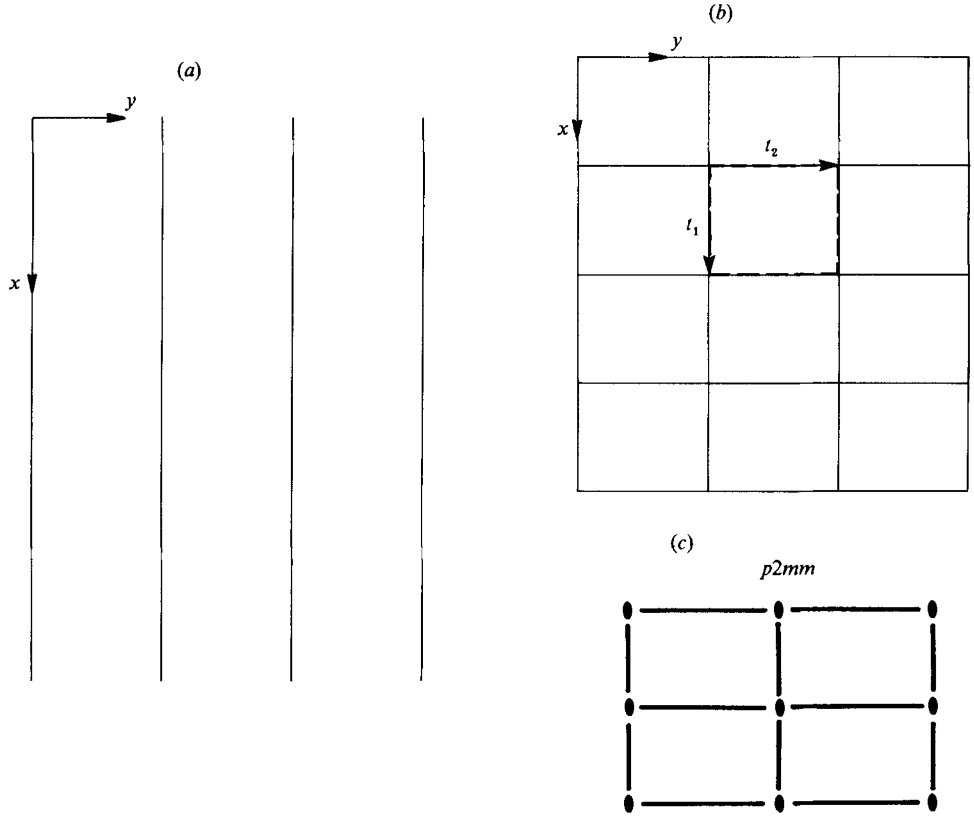

FIGURE 3. Transition from rolls (a) to a rectangular planform (b), called the cross-roll instability by Busse (1967b). The symmetry group of (b) shown in (c) results from the loss of the translational symmetry of (a). The transition is of type k.

invariant subgroups $p2mm$, with the location of m_x between 0 and t_3. The factor group is $Z_2 = \{E(p2mm), t_3(p2mm)\}$. As before $S_0 = T_0 = \eta_0 = \Gamma_1$, $S_1 = T_1 = \eta_1 = \Gamma_2$ with the symmetry group $p2mm$, $Ra_1 = 0$, and ψ_1 has the symmetry group $p2m'm'$. This transition was first investigated by Busse (1967b), and Busse & Whitehead's (1971) experiments suggest that the transition is a pitchfork bifurcation when the viscosity is constant. White (1981) carried out similar experiments in a fluid with $\eta = \eta(T)$. Though he did not determine the nature of the bifurcation, the argument above shows that it is not affected by the temperature dependence of the viscosity.

3.2. *t and k transitions in the absence of* C_{3z}

Most of the transitions that affected the square planform observed by White (1981, 1988) involved loss of a C_{4z} axis and were therefore t transitions. Several were then followed by k transitions, and one by a further t transition. In all cases studied by White, transitions involving the loss of C_{4z} (but not of C_{3z}, see §3.3) and another symmetry element occurred in two stages, each corresponding to the loss of a single element. The simplest of these transitions is that of cell stretching (White 1988, figures 13, 14), and is illustrated in figure 4.

The additional generator required to produce $p4mm$ from $p2mm$ is C_{4z}. It is first necessary to show that $p2mm$ is an invariant subgroup of $p4mm$ using the 2×2 matrices corresponding to the transformations on a plane. Then the order of the

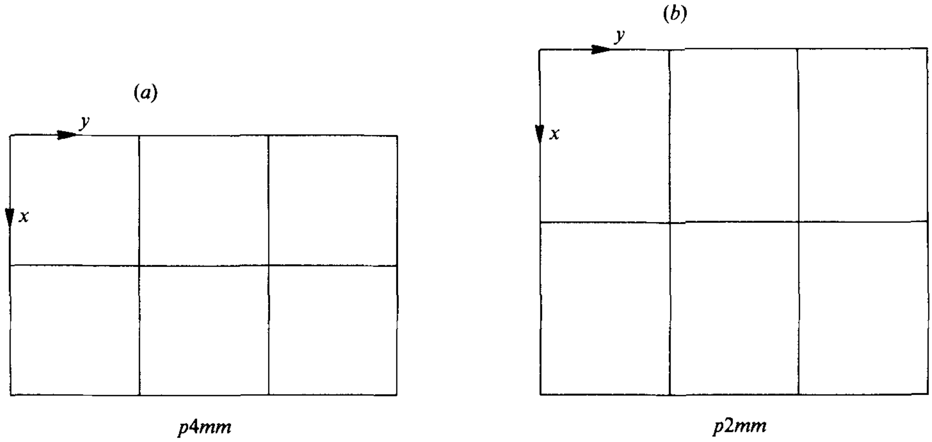

FIGURE 4. The square planform in (*a*) undergoes cell stretching in the x-direction to produce rectangles in (*b*). The symmetry element lost is C_{4z} and the transition is of type t.

factor group must be found by determining the number of independent cosets. Possible cosets are $E\{p2mm\}$, $C_{4z}\{p2mm\}$, $C_{4z}^2\{p2mm\}$ and $C_{4z}^3\{p2mm\}$. But $p2mm$ contains both the identity element and $C_{2z}(\equiv C_{4z}^2)$. Therefore the first and third possible cosets have an element, C_{2z}, in common and must be identical. The same argument applies to the second and the fourth, since $C_{4z}C_{2z} \equiv C_{4z}^3$. Hence there are only two cosets and the factor group is $Z_2 = \{E(p2mm), C_{4z}(p2mm)\}$. The arguments of §3.1 then show that $Ra_1 = 0$ and the transition is a pitchfork bifurcation.

White (1983) observed four different transitions that followed the path

$$p4mm \xrightarrow[C_{4z}]{t} p2mm.$$

The arrow in all such expressions points from the more symmetric to the less symmetric group, and does not indicate the direction in which the transition occurs. From the point of view of symmetry his cell splitting and chain instabilities, figure 5(*a* and *b*), are the same transition occurring in opposite directions. Cell splitting produces one new translational symmetry element (figure 5*a*) and therefore involves a transition to the minimal isomorphic supergroup of lowest index, whereas in the chain instability a translational element is lost (figure 5*b*) and the symmetry becomes that of the maximal isomorphic subgroup. In both cases the invariant subgroup is $p2mm$, the factor group is $Z_2 = \{E(p2mm), t_3(p2mm)\}$ and the transition is a pitchfork bifurcation.

A different type of k transition, which White (1988, figure 13*g*) called cell fusion, is illustrated in figure 6, and involves

$$p2mm \xrightarrow{k} c2mm.$$

The symmetry elements of $c2mm$ are illustrated in figure 6(*c*). The point-group symmetry of the two space groups is the same, and the translational elements are $t_1 = (a, 0)$, $t_2 = (0, b)$ and $t_3 = (a/2, b/2)$ for figure 6(*b*). The same elements together with $t_4 = (0, b/2)$ generate the symmetry group in figure 6(*a*), though the resulting unit cell is four times the size of the primitive cell and the generators are therefore different from those in IT (1983). The invariant subgroup of figure 6(*a*) is therefore

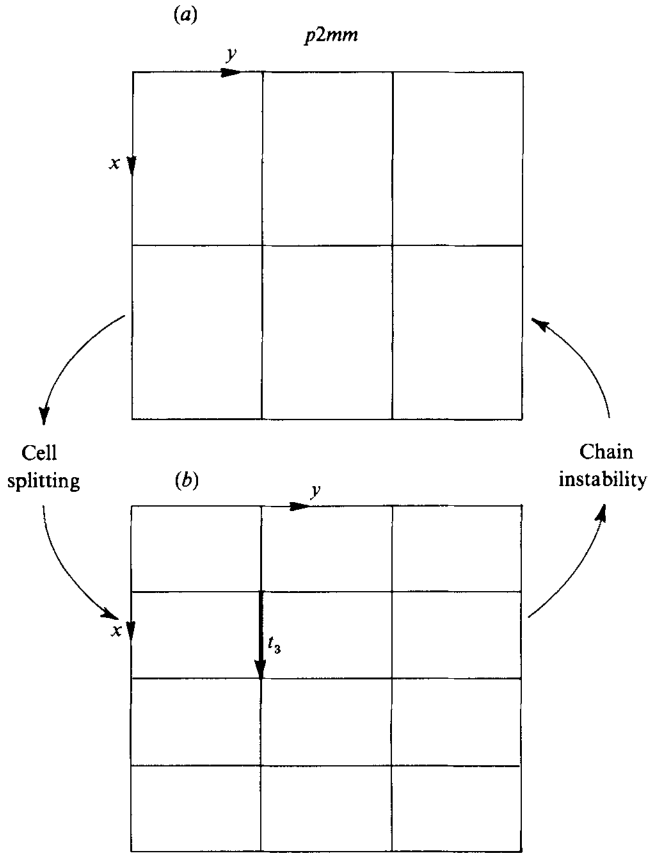

FIGURE 5. The symmetry element involved in the cell splitting and in the chain instabilities of White (1981, 1988) is the same. The transition from (*b*) to (*a*) is of type *k* and involves the loss of translation symmetry in the *x*-direction.

$c2mm$, the factor group is $Z_2 = \{E(c2mm), t_4(c2mm)\}$ and the transition is a pitchfork bifurcation with $Ra_1 = 0$.

The last type of instability that affected $p2mm$ in White's (1981) experiments he called the lip instability. The perturbed pattern has the symmetry group pm (figure 7*a*, *b*) and the transition occurs by loss of the C_{2z} element of $p2mm$.

Therefore

$$p2mm \xrightarrow[C_{2z}]{t} pm,$$

the factor group is $Z_2 = \{E(pm), C_{2z}(pm)\}$ and the bifurcation is again a pitchfork with $Ra_1 = 0$.

White (1988, figure 14) also observed a change in planform from $p2mm$ to one with that of a point group $2mm$ and no periodic symmetry elements. A similar transition is discussed in §3.3, where the transition occurs to an isomorphic subgroup other than the maximal isomorphic subgroup. Such transitions are nonetheless *k* transitions and lead to pitchfork bifurcations.

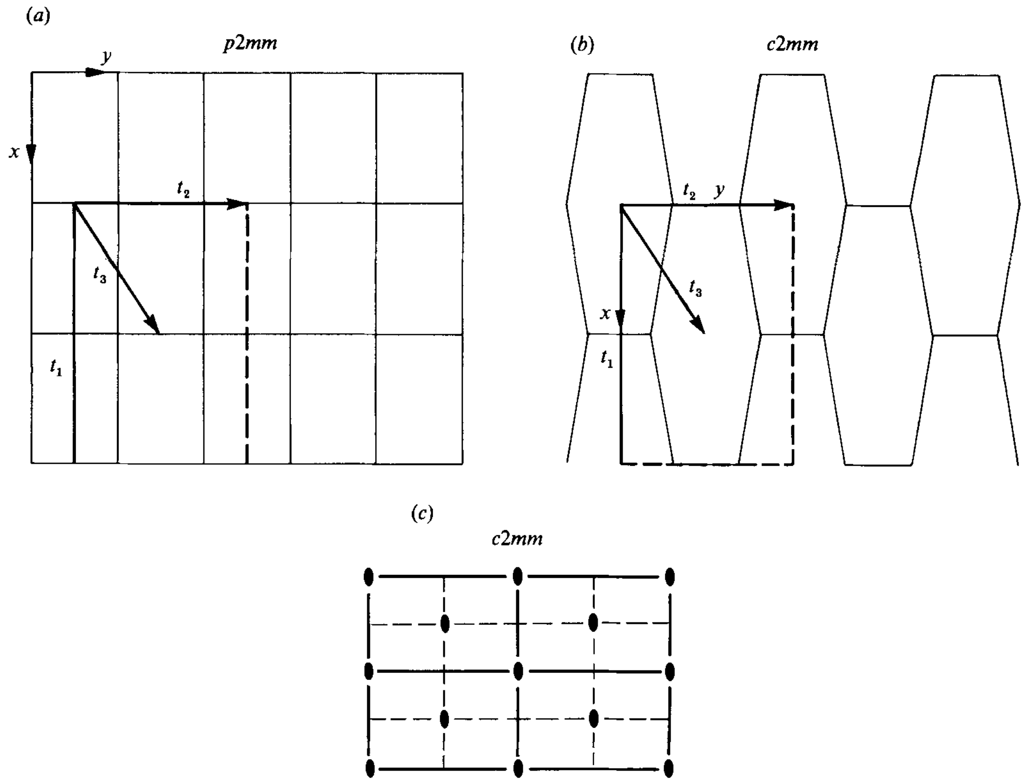

FIGURE 6. The cell-fusion instability of White (1988) involves a k-type transition in which a translational symmetry element in the x- and y-directions are lost, but the centring symmetry element t_3 in (a) is not. The transition is of type k, and is to the plane group $c2mm$, (c).

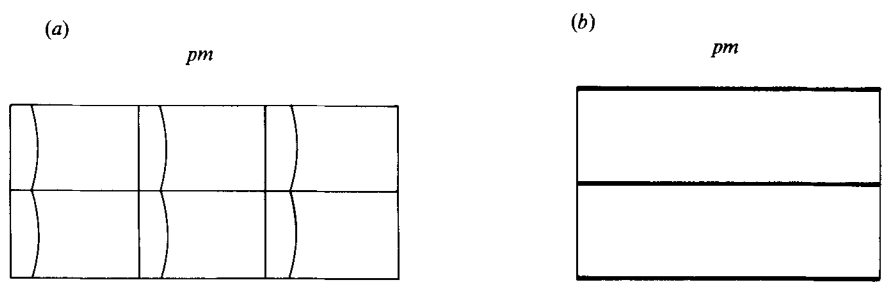

FIGURE 7. White's (1981) lip instability changes the planform from the rectangular planform $p2mm$ in figure 3(b) to pm in (a), with mirror planes in (b), by loss of C_{2z}, and is therefore a t transition.

3.3. *t and k transitions associated with* C_{3z}

Transitions involving planforms with C_{3z} (or C_{6z}) symmetry elements are more difficult to analyse than are those with C_{4z} and C_{2z} because the subgroup $\{E, C_{3z}, C_{3z}^2\}$ interacts in a complicated way with other symmetry elements. Therefore the invariant subgroups are not the symmetry groups of the less symmetric patterns. For instance, inspection of figure 1 shows that $2mm$ is a subgroup, but not an invariant subgroup, of $6mm$. Hence the identity element of the factor group cannot be $E\{2mm\}$. White studied various types of transition that occur in patterns of up and down hexagons, labelled by the direction of the flow on the C_{6z} axis. White referred to down hexagons as triangles. From the point of view of planform transitions the symmetry of up and down hexagons is the same.

The simplest transition of the hexagonal planform to analyse is that illustrated in figure 8, where the two opposite sides of each hexagon become longer and the other four shorter. This transition was also studied by Richter (1978)

$$c6mm \xrightarrow[C_{3z}]{t} c2mm.$$

The unit cell in figure 8(*a*) is not the primitive unit cell used in IT (1983), which is shown as a dotted rhombus. The advantage of using the centred lattice c is that the x- and y-axes can be chosen to be orthogonal, and the same cell is common to $c6mm$ and $c2mm$. The generators for $c6mm$ are

$$t_1 = (2\sqrt{3}, 0), \quad t_3 = (\sqrt{3}, 3), \quad C_{3z}, \quad C_{2z}, \quad m_y, \tag{3.9}$$

and $t_2 = t_3^2 t_1^{-1}$. The length of the side of each hexagon is 2 units. The generators for $c2mm$ are t_1, t_3, C_{2z} and m_y. The obvious subgroup to choose is $c2mm$, which must be tested to discover if

$$C_{3z}\{c2mm\}\, C_{3z}^2 = \{c2mm\}. \tag{3.10}$$

Equation (3.10) is true for all generators except m_y, for which

$$C_{3z}\, m_y\, C_{3z}^2 = C_{3z}^2\, m_y. \tag{3.11}$$

The right-hand side is not an element of $c2mm$. Equation (3.11) may also be written

$$(C_{3z}^2\, m_y)^2 = E. \tag{3.12}$$

Therefore $c2mm$ is not an invariant subgroup. If, however, $c2$ is used instead it is straightforward to show that

$$C_{3z}\{c2\}\, C_{3z}^2 = \{c2\}, \tag{3.13}$$

$$m_y\{c2\}\, m_y = \{c2\}. \tag{3.14}$$

Hence $c2 \lhd c6mm$, $c2 \lhd c2mm$ and $c6mm$ may be written

$$\{c6mm\} = \{c2\} + C_{3z}\{c2\} + C_{3z}^2\{c2\} + m_y\{c2\} + m_y C_{3z}\{c2\} + m_y C_{3z}^2\{c2\}. \tag{3.15}$$

These cosets are all different, and therefore the order of the transition group is 6. However the second two and the last three cosets are conjugate to each other, because (3.12) shows that

$$\left.\begin{aligned} m_y C_{3z}^2 m_y &= C_{3z}, \\ C_{3z}^2(m_y C_{3z})\, C_{3z} &= m_y, \\ C_{3z}^2(m_y C_{3z}^2)\, C_{3z} &= m_y C_{3z}. \end{aligned}\right\} \tag{3.16}$$

Therefore the factor group $\{c6mm\}/\{c2\}$ is isomorphic to D_3

$$\left.\begin{array}{lcccc} \boldsymbol{D}_3 & E(c2) & C_{3z}, C_{3z}^2 & m_y, m_y C_{3z}, m_y C_{3z}^2 & \\ \Gamma_1 & 1 & 1 & 1 & c6mm \\ \Gamma_2 & 1 & 1 & -1 & c6 \\ \Gamma_3 & 2 & -1 & 0 & c2mm, \end{array}\right\} \tag{3.17}$$

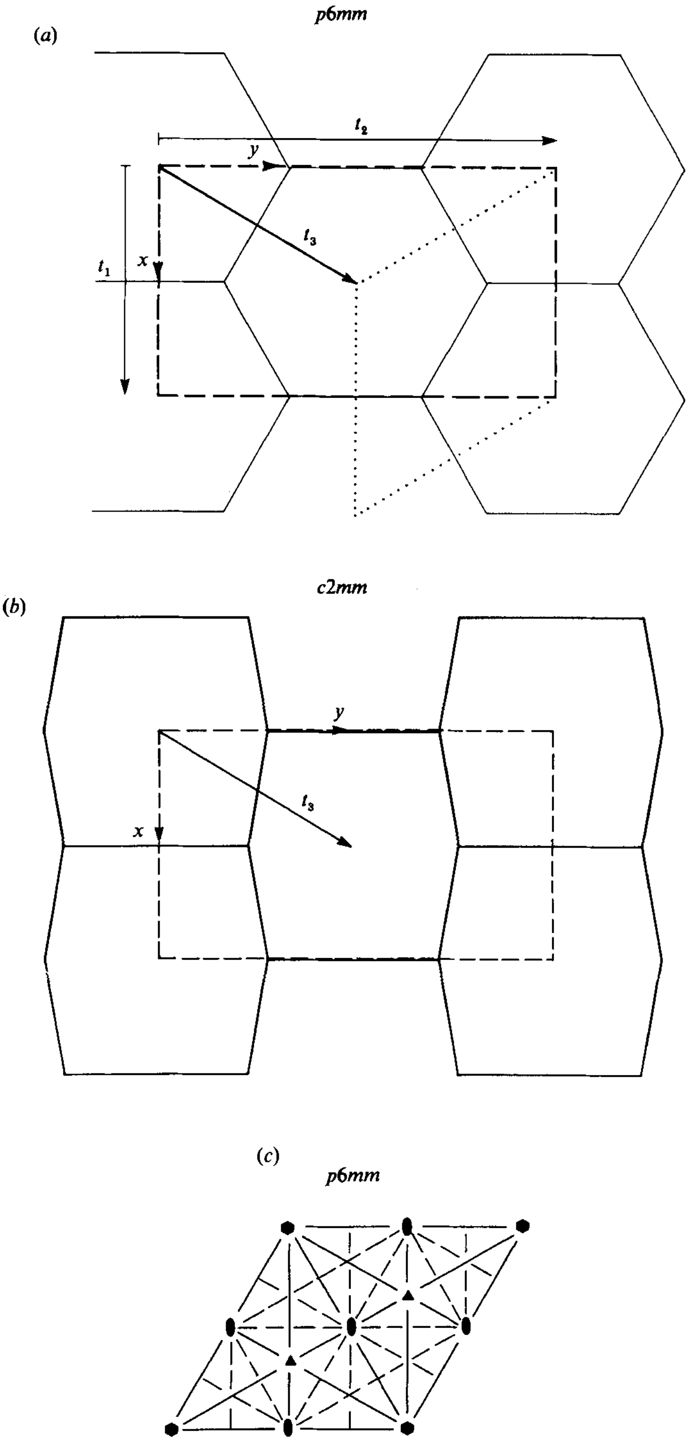

FIGURE 8. The transition from hexagons (a) to a planform like that in (b) involves the loss of C_{3z} and is therefore of type t. The conventional unit cell in (a) is shown dotted, with the symmetry elements in for p6mm (c). It is, however, more convenient to use a larger unit cell, c6mm, with orthogonal axes, shown by the dashed lines in (a), which is common to both p6mm and c2mm in (b).

where for Γ_3

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad C_{3z} = \frac{1}{2}\begin{pmatrix} -1 & -\surd 3 \\ \surd 3 & -1 \end{pmatrix}, \quad m_y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.18}$$

This factor group differs from all those previously considered in having a two-dimensional irreducible representation, which is also a faithful representation of the group. The relevant symmetry equations are (3.7), (2.56), (2.62)–(2.65). As is the case for all planform transitions, $\hat{z} = T_0 = \Gamma_1$. $(\hat{x} \wedge \hat{y})$ changes sign under m_y, but not under C_{3z}. Therefore, $(\hat{x} \wedge \hat{y}) = \Gamma_2$. It is also clear that $T_1 = \Gamma_3$, because it cannot be invariant under C_{3z}. The character table (3.17) shows that $\Gamma_2 \times \Gamma_3 = \Gamma_3$, and therefore that the only term in (2.63) that transforms as does Γ_1 is that containing T_0. Hence the transition is a pitchfork bifurcation.

The existence of a two-dimensional representation results in a number of complications. The factor group (3.17) can describe transitions to both *c*2 and to *c*2*mm*, and both of these perturbations must transform like Γ_3. There is therefore a degeneracy, and the critical Rayleigh number for the transition to either space group must be the same. For this reason it is necessary to choose the correct basis functions to describe the transition. Such functions can be represented by a vector $\boldsymbol{a} = (a_1, b_1)$, where a_1 and b_1 are arbitrary constants. The representation of m_y shows that the general basis vector corresponds to *c*2, whereas a transition to the space group *c*2*mm* requires $b_1 = 0$. Since a_1 is arbitrary, the basis vector can be written $\boldsymbol{a} = (1, 0)$. Another difference is that $\Gamma_3^2 = \Gamma_1 + \Gamma_2 + \Gamma_3$, and hence the nonlinear interactions can lead to complicated circulations involving all three representations. One further difference concerns the definition of the projection operators, and is discussed in the Appendix.

The relationship between rolls, $p_{01}\,2mm$, and *c*2*mm* is the same as that between rolls and *p*2*mm* discussed in §3.1. Therefore the factor group is Z_2 and the bifurcation is a pitchfork. These bifurcations can be written

$$c6mm \xrightarrow[C_{3z}]{t} c2mm$$

$$p_{01}\,2mm \xrightarrow{k} c2mm.$$

Another transition White (1981) observed involved a change from *p*6*mm* to the planform in figure 9(*a*). This plane group is *p*2*mg* and is non-symmorphic. The origin of the standard unit cell for this group in IT (1983) has been moved to make it the same as that in figure 8(*a*), and the *x*- and *y*-axes have been interchanged. It is convenient to use Hermann's rules and to consider the transition

$$c6mm \rightarrow p2mg$$

as consisting of two parts: $$c6mm \xrightarrow[C_{3z}]{t} c2mm,$$

$$c2mm \xrightarrow{k} p2mg.$$

As in figure 8 the generators of *c*2*mm* are E, t_1, t_3, C_{2z}, m_y with C_{2z} and m_y at the origin of the coordinate system. Neither t_3 nor C_{2z} in this position is a generator for

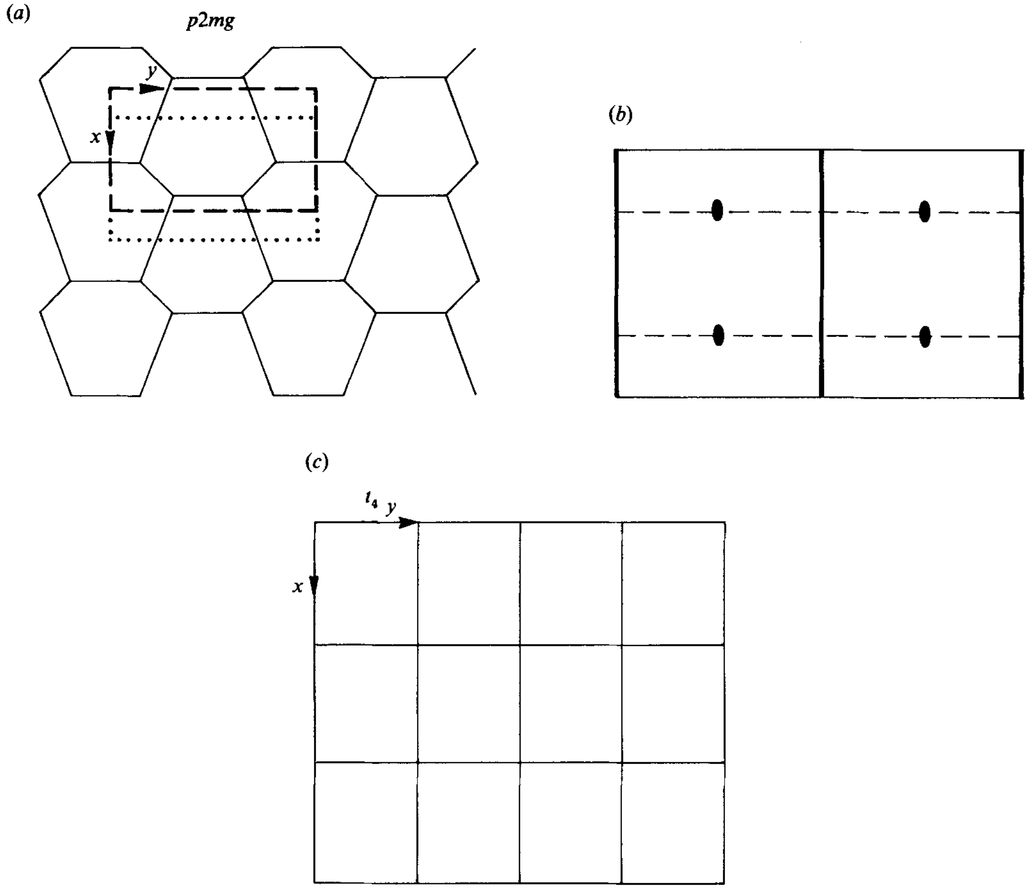

FIGURE 9. The transition of hexagons to rectangles involves an intermediate planform in which every alternate column of hexagons is shifted, to produce the planform $p2mg$ in (a) with symmetry elements in (b). This non-symmorphic group is an invariant subgroup of $c2mm$ and $p2mm$ shown in (c). The transition is of type k. The dashed unit cell is used to describe the transition from figure 8(b), the dotted to that in figure 9(c).

$p2mg$. Instead the C_{2z} axis is at $t_3/2$ and will be denoted C^*_{2z}, and the generators of $p2mg$ are E, t_1, C^*_{2z}, m_y. The symmetry positions generated by $c2mm$ on (x, y) are

$$\left.\begin{array}{cccc} E, & C_{2z}, & m_y, & m_y C_{2z} (\equiv m_x) \\ (x, y) & (\bar{x}, \bar{y}) & (x, \bar{y}) & (\bar{x}, y) \\ t_3, & t_3 C_{2z}, & t_3 m_y & t_3 m_y C_{2z} \\ (x+\sqrt{3}, y+3) & (\bar{x}+\sqrt{3}, \bar{y}+3) & (x+\sqrt{3}, \bar{y}+3) & (\bar{x}+\sqrt{3}, y+3), \end{array}\right\} \tag{3.19}$$

where those for $p2mg$ are

$$\left.\begin{array}{cccc} E, & C^*_{2z}, & m_y, & m_y C^*_{2z} \\ (x, y) & (\bar{x}+\sqrt{3}, \bar{y}+3) & (x, \bar{y}) & (\bar{x}+\sqrt{3}, y-3). \end{array}\right\} \tag{3.20}$$

Clearly $p2mg$ is a common subgroup, and it is easy to show that it is invariant. Then

$$\{c2mm\} = \{p2mg\} + t_3\{p2mg\}. \tag{3.21}$$

Therefore the factor group $\{c2mm\}/\{p2mg\}$ is isomorphic with Z_2, $Ra_1 = 0$ and the transformation is a pitchfork bifurcation.

The $p2mg$ pattern in figure 9(*a*) was observed by White (1981) to change to the rectangular planform in figure 9(*c*), which is $p2mm$ and symmorphic. The generators of $p2mm$ are t_1, t_2, C_{2z} and m_y through $(0,0)$. If the same unit cell is used for $p2mg$ and $p2mm$ another element, $t_4 = (0,3)$, must be included, since there are two primitive $p2mm$ cells to each primitive $p2mg$. Furthermore the unit cell for $p2mg$ shown in figure 9(*b*) must be displaced by $(\sqrt{3}/2, 0)$ to be that shown dotted in figure 9(*a*), so that the C_{2z} axes of $p2mm$ and $p2mg$ coincide. Then $p2mg$ is the invariant subgroup, and

$$\{p2mm\} = \{p2mg\} + t_4\{p2mg\}. \tag{3.22}$$

This transition is therefore a pitchfork bifurcation with $Ra_1 = 0$. This series of transitions may therefore be written

$$\begin{gathered} c3mm \xrightarrow[C_{3z}]{t} c2mm, \\ c2mm \xrightarrow{k} p2mg, \\ p2mm \xrightarrow{k} p2mg. \end{gathered} \tag{3.23}$$

Another transition that White observed in his study of hexagons is illustrated in figure 10(*b*). After the loss of C_{3z} to produce $c2mm$, an enlargement of the unit cell occurred, by a factor of five in the x-direction. This transition is a k transition. The generators of figure 10(*a*) are

$$t_1 = (2\sqrt{3}, 0), \quad t_3 = (\sqrt{3}, 3), \quad m_y, \quad C_{2z}, \tag{3.24}$$

with $t_2 = t_3^2 t_1^{-1}$, and those of figure 10(*b*) are

$$t_4 = (10\sqrt{3}, 0) = 5t_1, \quad t_5 = (5\sqrt{3}, 3), \quad m_y, \quad C_{2z}, \tag{3.25}$$

with $t_2 = t_5^2 t_4^{-1}$. This transition is therefore not to the maximal isomorphic subgroup of lowest index, which would have

$$t_4 = (6\sqrt{3}, 0) = 3t_1, \quad t_5 = (3\sqrt{3}, 3).$$

It is straightforward to show that

$$\left.\begin{aligned} t_1^4 C_{2z} t_1 &= t_1^3 C_{2z}, \\ \text{or} \qquad C_{2z} t_1 &= t_1^4 C_{2z}. \end{aligned}\right\} \tag{3.26}$$

Therefore the invariant subgroup cannot contain C_{2z}, but can contain m_y (but not m_x). The correct choice is therefore cm, and, because of (3.26), this leads to a factor group that is isomorphic to D_5:

D_5	$E(cm)$	t_1, t_1^4	t_1^2, t_1^3	$C_{2z}, t_1 C_{2z}, t_1^2 C_{2z}$ $t_1^3 C_{2z}, t_1^4 C_{2z}$	
Γ_1	1	1	1	1	$c2mm(a)$
Γ_2	1	1	1	-1	$cm(a)$
Γ_3	2	$2\cos\beta$	$2\cos 2\beta$	0	$c2mm(b)$ $cm(b)$
Γ_4	2	$2\cos 2\beta$	$2\cos\beta$	0	$c2mm(b)$ $cm(b)$,

(3.27)

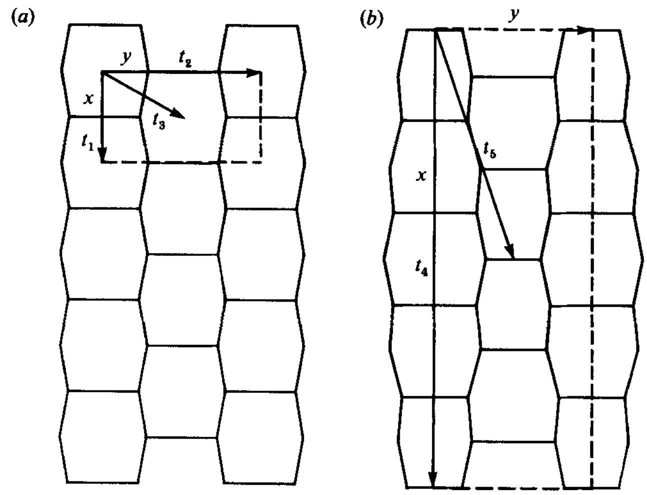

FIGURE 10. A transition that occurs in the *c2mm* planform produced from the breakdown of hexagons increases the size of the unit cell in (*a*), shown dashed, by a factor of 5 to produce the planform in (*b*). This transition is of type *k* but is not to the maximal isomorphic subgroup, which would only increase the unit cell size by a factor of 3.

where the *a* and *b* in brackets following the space group refer to figure 10,

$$\beta = 2\pi/5 = 72^{\circ}$$

and the two-dimensional matrices representing the generators are

$$t_1 = \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix}, \quad C_{2z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.28}$$

for Γ_3 and

$$t_1 = \begin{pmatrix} \cos 2\beta & \sin 2\beta \\ -\sin 2\beta & \cos 2\beta \end{pmatrix}, \quad C_{2z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.29}$$

for Γ_4. Both two-dimensional representations have the correct transformational behaviour to represent T_1, and it is not obvious whether one or both are required. The perturbation which transforms in the same way as does Γ_3 need not have the same critical Rayleigh number as that which transforms like Γ_4. The group D_5 most commonly occurs as a point group with elements C_{5z} and C_{2x}, which is clearly isomorphic to D_5. The element C_{5z} can then be represented by the appropriate rotation matrix, which generates a rotation through an angle $-\beta$. Since the representation Γ_3 contains the same matrix as the representation of one of the generators, it must correspond to such a rotation. All conventional perturbations are invariant when rotated through 2π, a rotation corresponding to -5β and the unit element of D_5. They are therefore represented by Γ_3. In the representation Γ_4, however, C_{5z} corresponds to a rotation through an angle -2β. Therefore any property described by this representation changes sign when rotated through 2π, and is only invariant when rotated through 4π. No conventional physical quantities transform in this manner. Only spin in quantum mechanics behaves in this way, and therefore representations such as Γ_4 are known as spin representations. They are therefore not to be expected in fluid mechanics. Somewhat surprisingly, this argument is incorrect. When the general Fourier expansion of *c2mm(b)* is projected onto Γ_1, Γ_3 and Γ_4 (see Appendix), one-fifth of the terms transform as does Γ_1, two-

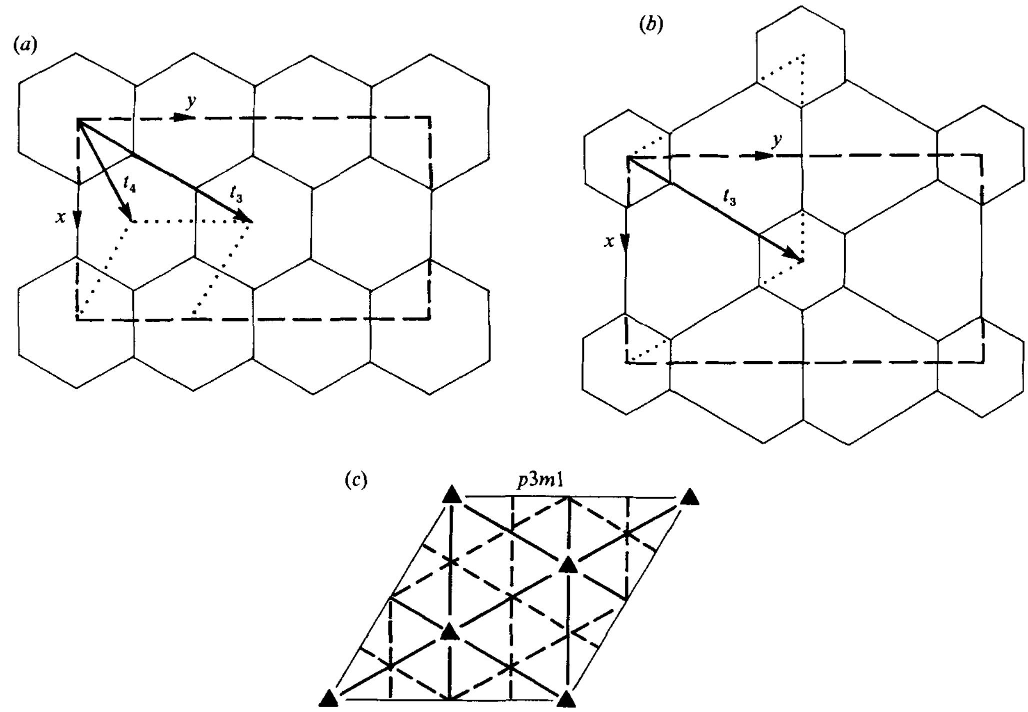

FIGURE 11. The mosaic instability with an hexagonal planform increases the unit cell by a factor of 3 with the primitive cells shown dotted in (a) and (b). The $c6mm$ unit cell of (b) is shown dashed in (a) and (b). The transition involves the loss of the translational symmetry element t_4 in (a), but the symmetry group of (b) is not an invariant subgroup of that of (a). The largest invariant subgroup that is invariant and common to both is $c3m1$, with the symmetry elements for the primitive cell $p3m1$ in (c).

fifths as does Γ_3 and two-fifths as does Γ_4. Therefore instabilities that transform as do Γ_3 and Γ_4 are allowed.

White (1988, figures 16 and 17) proposed two other transitions from hexagons. He observed only one of these: the mosaic instability in hexagons illustrated in figure 11. Though this instability resembles the mosaic instability in squares illustrated in figure 2, the factor group is not Z_2 and the area of the unit cell increases by a factor of 3 instead of 2. The mosaic transition in hexagons is clearly of type k, since both planforms are $p6mm$ (drawn as usual as $c6mm$). The primitive unit cell of figure 11 (b) is shown dotted. The generators are

$$\left.\begin{array}{l} t_1 = (6,0), \quad t_3 = (3, 3\surd 3), \quad C_{2z}, \\ C_{3z}, \quad m = \dfrac{1}{2}\begin{pmatrix} 1 & -\surd 3 \\ -\surd 3 & -1 \end{pmatrix}, \end{array}\right\} \tag{3.30}$$

with $t_2 = t_3^2 t_1^{-1}$. This choice of generators allows m, reflection in a plane at right angles to t_3, to be an element of the invariant subgroup. The additional generator needed for figure 11 (a) is $t_4 = (3, \surd 3)$. Inspection of figure 11 (a) shows that $t_4^3 = t_1 t_3$. The obvious subgroup to test for invariance is therefore $c6mm(b)$. It is not an invariant subgroup because

$$t_4^{-1} C_{2z} t_4 = t_4 C_{2z}. \tag{3.31}$$

Since the cosets of whatever invariant subgroup of $c6mm(a)$ is used must contain t_4, no such subgroup can contain C_{2z}. The largest possible subgroup is then $c3m1$, and it is easy to show that this group is indeed an invariant subgroup of $c6mm(a)$ using rotation matrices. Then the factor group is D_3 and

$$\{c6mm(a)\} = \{c3m1\} + t_4\{c3m1\} + t_4^2\{c3m1\} \\ + C_{2z}\{c3m1\} + t_4 C_{2z}\{c3m1\} + t_4^2 C_{2z}\{c3m1\}. \quad (3.32)$$

As before the last three cosets and the second two are conjugate to each other, and the character table is (3.17). All operators, T_0, S_0, ψ_0 and η_0 transform as does Γ_1. T_0, S_0 and η_0 have the symmetry group $c6mm(a)$ whereas ψ_0 has symmetry $c6m'm'(a)$. T_1, S_1, η_1 and $(\hat{x} \wedge \hat{y}) \times \psi_1$ transform like Γ_3, with a basis vector $(1,0)$. Therefore $Ra_1 = 0$ and the transition is a pitchfork bifurcation. Unlike (3.17), there is no obvious geometric interpretation of the matrices corresponding to Γ_3.

The other transition proposed by White (1988, figure 17) he called cell fusion in hexagons, though he did not observe it in his experiments. The transitions involved resemble those in the hexagonal mosaic instability, but are more complicated. The transitions he sketches are from $c6mm$ to $c31m$, and consist of both a t and a k transition, followed by one to $c6mm$. Using Hermann's rules these can be written as

$$\left.\begin{array}{l} c6mm \xrightarrow[C_{2z}]{t} c3m1 \\ c3m1 \xrightarrow{k} c31m \\ c6mm \xrightarrow[C_{2z}]{t} c31m. \end{array}\right\} \quad (3.33)$$

The first transition involves a factor group Z_2 with elements $E\{c3m1\}$, $C_{2z}\{c3m1\}$. The second increases the size of the unit cell by a factor of 3 through the loss of t_4. The factor group is D_3 and is given by (3.32), with C_{2z} replaced by m and $c3m1$ by $c3$. Both generators C_{3z} and m must be chosen in a different way from (3.32). The final transition is a t transition with Z_2 as the factor group. The transition (3.33) is therefore more complicated than any of those discussed above. Since White did not observe this transition, it is not discussed here in detail.

3.4. *General remarks about planform transitions when $\eta = \eta(T)$*

The considerable variety of transitions analysed in the last three sections and listed in table 2 are all pitchfork bifurcations. The observed behaviour therefore suggests that such transitions are preferred. All factor groups also contain Z_2, either as the factor group itself, or as the invariant subgroup in a semidirect product with Z_3 or Z_5.

Transitions of type t leave the area of the unit cell unchanged. Because there are only a small finite number of point symmetry elements in any planform, the number of possible t transitions is limited. In contrast a doubly infinite number of k transitions are possible in an infinite plane group. The symmetry element that is lost is a translation, which is often in a direction not parallel to either the x- or y-axis. Two observations suggest that the critical Rayleigh numbers for k transitions to subgroups other than the maximal subgroup are not very different from that to the maximal subgroup. The maximal k subgroup of the pattern in figure 10(a) increases the length of the side of a unit cell by a factor of 3 in the x- or y-direction, whereas the transition observed by White was to the pattern in figure 10(b) which did so by a factor of 5. The other observation concerns a transition White observed from $p4mm$ to a pattern with a $2mm$ symmetry axis, but with no periodicity in either the x- or y-direction. Since White's photograph (White 1988, figure 14c) extends for 5 unit cells of $p2mm$ in each direction, the occurrence of this instability suggests that the Rayleigh number for the maximal subgroup of $p2mm$, produced by a t transition from $p4mm$, is little different from that of other $p2mm$ patterns with much larger unit cells.

4. Transitions involving symmetry in the z-direction

4.1. *Marginal stability*

More work has been carried out on marginal stability than on any other problem in thermal convection, because standard perturbation theory can be applied. A full discussion of the problem is complicated because the space group of the initial conductive solution is a Lie group, of which the perturbation is a discrete, but not an invariant, subgroup. The factor group required is therefore infinite, and contains infinitely many two-dimensional representations. The discussion below is therefore incomplete, though the results are correct. A full treatment will be published elsewhere (D. McKenzie, in preparation).

As is always the case, the question of which planform will be preferred depends on the magnitude of the coefficients in the equations, and cannot be investigated using factor groups. But the nature of the transition can be, and group theory was used for this purpose by Golubitsky *et al.* (1984). Probably the best known of all pitchfork bifurcations is that originally studied by Rayleigh: the onset of convection in a layer with stress-free boundaries. The planform he investigated consisted of rolls, but hexagons with the same wavenumber can be produced by superimposing three rolls at angles of 60°. Rolls without a symmetry element in the vertical plane, for instance with a rigid upper boundary and a stress-free lower boundary (Schlüter, Lortz & Busse, 1965), or with a viscosity that is a function of temperature (Busse 1967a), are also formed by a pitchfork bifurcation. However, Busse (1967a) showed that the transition becomes a transcritical bifurcation when the planform is hexagonal and the viscosity is a function of temperature.

The first example is that of two-dimensional rolls in the (y, z)-plane illustrated in figure 12. The perturbed temperature in figure 12(b) has symmetry group $a2'mm'(c2'm'm)$ and translational symmetry elements t_1^2, t_2^2, and $t_1 t_2$. For the reasons discussed in §2, the order of the factor group involved in any transition is reduced if the symmetry group of $\hat{z} \times T$, $a2mm$, rather than that of T, is used in the analysis. The boundary conditions on $z = 0, 1$ are the same and η is constant. $\hat{z} \times T_0$ (figure 12a) has all the symmetry elements of $a2mm$, and in addition is unchanged under t_3 and any vector in the y-direction. The important feature of the plane group in figure 12(b) is that it has no symmetry elements, either ordinary or black and white, which are not also possessed by $\hat{z} \times T_0$. In particular, as figure 12(c) shows, it is not invariant under t_2'. Choosing t_2 as the active element and $a2mm$ as the invariant subgroup produces a factor group Z_2

$$\left.\begin{array}{ccc} Z_2 & E(a2mm) & t_2 \\ \Gamma_1 & 1 & 1 \\ \Gamma_2 & 1 & -1 \end{array}\right\}, \tag{4.1}$$

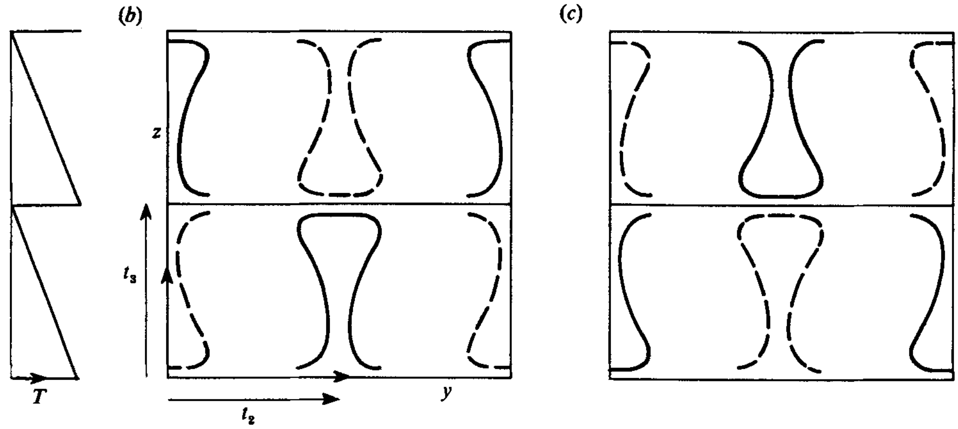

FIGURE 12. (a) Shows the conductive solution $T_0(z)$, which is independent of y. (b) Represents a sketch of the isotherms of T_1 in the vertical plane for a roll solution, when the nonlinear terms are important, with hot regions shown as continuous, cold regions as dashed, isotherms. The operation $\mathbb{R}$ changes the sign of the temperature, to produce the pattern shown in (c). The symmetry group of (b) is $a2'mm'(c2'm'm)$.

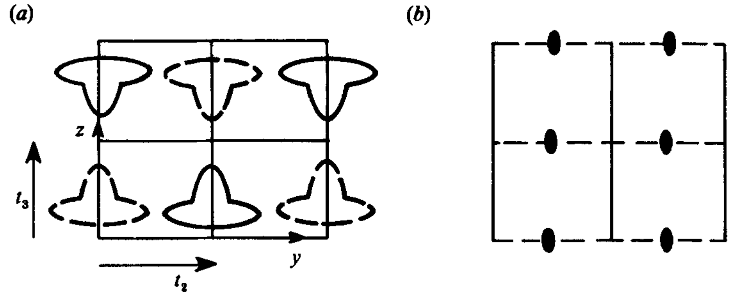

FIGURE 13. (a) A sketch of the isotherms of rolls in the vertical plane, like that in figure 12, but when the nonlinear terms are unimportant and when perturbation theory can be used. The centring symmetry element in figure 12 is absent, but has been replaced by a black and white symmetry element t_2'. (b) Shows the symmetry elements of $p2mg$, the ordinary space group of (a).

$\hat{z}\times T_0 = \Gamma_1$. If $\hat{z}\times T_1$ transformed as does Γ_2 its sign would be changed by t_2 and therefore it would have a black-and-white symmetry element t_2'. It clearly does not, and therefore must be represented by both Γ_1 and Γ_2. The transition is therefore transcritical. This surprising result contradicts both theory and experiment. The cause of the contradiction lies in the shape of the isotherms sketched in figure 12, which is correct for vigorous convection, but has a lower symmetry than that determined from first-order perturbation theory. At this order the temperature eigenfunction for rolls is $f(z)\cos(2\pi y/\lambda)$, where $\lambda = 2t_2$. Unlike the pattern in figure 12, this function does have a symmetry element t_2'.

A sketch of such a function in figure 13 has been drawn with no mirror planes m_z at $(0,0,t_3/2)$ and $(0,0,3t_3/2)$, and transforms as does Γ_2. The invariant subgroup of $\hat{z}\times T_0$ and $\hat{z}\times T_1$ is $a2mm$, and the factor group is $Z_2 = \{E(a2mm), t_2(a2mm)\}$. The transition is therefore a pitchfork bifurcation. The same result also applies to the roll instability when $\eta = \eta(T)$. The invariant subgroup is then the line group $\not{p}m$ and $\eta_0 = \Gamma_1$, $\hat{z}\times T_1$, $\hat{z}\times S_1$ and $\eta_1 = \Gamma_2$.

When the marginally stable planform is hexagonal the problem is more complicated because t_2' is no longer a symmetry element. If the viscosity is constant,

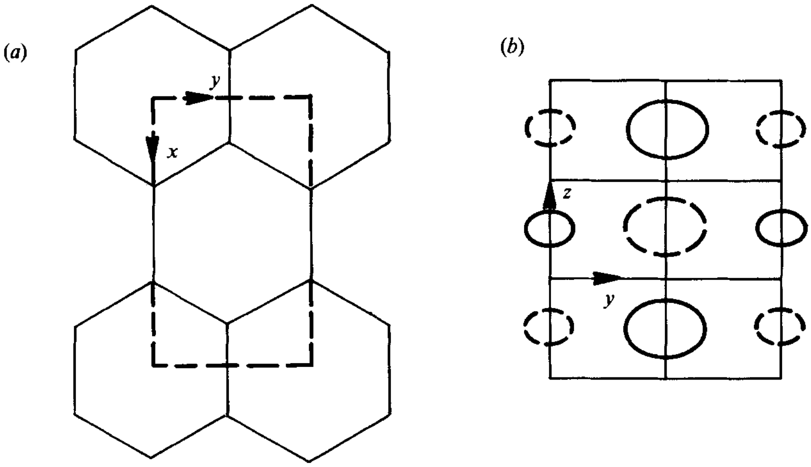

FIGURE 14. The planform (*a*) and isotherms (*b*) in a section in the (y, z)-plane through $x = 3$, the middle of the unit cell. The boundary conditions are the same on $z = 0, 1$ and the fluid is taken to be Boussinesq. The isotherms illustrated are those of the perturbation solution, and have a mirror plane m_z through $z = 1/2$.

and if the boundary conditions are the same on $z = 0, 1$, then T_1 and S_1 are unchanged by a reflection m_z through $z = 1/2$. In contrast T_0 has a symmetry element m_z' through this point. Figure 14 shows that the pattern in the layer $1 \geqslant z \geqslant 0$ continued in the vertical direction by reflection m_z' through $z = 0$ and 1. Because of the difference in the symmetry groups of T_0 and T_1, this continuation leaves T_0 unchanged but reverses the sign of T_1. The resulting pattern shown in figure 14 demonstrates that T_1 possesses a symmetry element t_3' that is not shared by T_0. The invariant subgroup of $\hat{z} \times T_0$ is therefore $C6/mmm$ and the factor group is $Z_2 = \{E(C6/mmm), t_3(C6/mmm)\}$ with $\hat{z} \times T_0 = \Gamma_1$ and $\hat{z} \times T_1 = \Gamma_2$. Then $Ra_1 = 0$ and the transition is a pitchfork bifurcation. When the viscosity is constant and the boundary conditions are not the same on $z = 0, 1$, $\hat{z} \times T_1$ and $\hat{z} \times S_1$ no longer possess m_z as a symmetry element. The perturbed solution then no longer has a symmetry element that is not present in T_0 and the bifurcation would be expected to be transcritical, as Malkus & Veronis (1958) argued. In fact this is not the case, because the linear operators in the governing linearized equations are self-adjoint (Schlüter *et al.* 1965). This property does not have any obvious relationship to the symmetry groups of T_0 and T_1. In contrast to the symmetry arguments, which are independent of whether the equations are linear or nonlinear, self-adjointness is a property of linear equations only. It is therefore not surprising that symmetry arguments fail to show that the transition is a pitchfork bifurcation. In general a sufficient, but not a necessary, condition for a transition to be a pitchfork bifurcation is that $Ra_1 = 0$ because of symmetry. As the example above illustrates, Ra_1 may be zero when it is not required to be by symmetry. This special property of the linearized equations governing marginal stability reduces the usefulness of symmetry arguments in such problems. Such arguments are most useful when the nonlinear terms are dominant, since few other approaches are then available. But it is nonetheless important to remember that, though the symmetry may be insufficient to demonstrate that $Ra_1 = 0$, this condition may still be satisfied.

The last case to consider is the onset of convection when $\eta = \eta(T)$, which Busse

(1967*a*) showed to be a transcritical bifurcation when the planform was hexagonal. Equation (2.63) contains η_0, which, as (2.64) shows, is not symmetric or antisymmetric with respect to a change in sign of T_0. There are therefore no translation symmetry elements in any direction, and the transition is expected to be transcritical. In this example this result is correct. Since the same argument applies to the transition to the square planform from the conductive solution when the viscosity depends on temperature, this may also occur through a transcritical bifurcation.

All the examples in which a pitchfork bifurcation could be shown to occur because of symmetry have a perturbed temperature that has a plane or a space group containing a symmetry element t_2' or t_3' that is not present in T_0. Instead T_0 has symmetry elements t_2 and t_3. Therefore all these plane and space groups are examples of type IV Shubnikov groups, where the black and white operation is a translation. When this condition is not satisfied by the perturbation to $O(\epsilon)$, as in figure 12(*b*), the transition may be transcritical. In figure 12(*b*) the black and white symmetry element is associated with rotation. The group is therefore a type III Shubnikov group, all of whose symmetry elements are also possessed by T_0. It is commonly argued that the two solutions that originate from the pitchfork bifurcation at marginal stability correspond to that shown in figure 13(*a*), and to one shifted by t_2 to the right. This view is only correct when $\mathbb{R}t_2$ is a symmetry element of T_1, and even then is misleading. When this condition is satisfied the operation $\mathbb{R}$ on T_1 leads to the same pattern as does t_2, and therefore the effect of the two operations is the same. But it is $\mathbb{R}$, not t_2, that is the important element, because $\mathbb{R}T_1$ is a solution whereas $\mathbb{R}T_0$ is not. The hexagonal pattern in figure 14 illustrates this distinction. When the viscosity is constant, however, the marginal stability problem is complicated by being governed by differential equations that are self-adjoint when they are linearized. Hence the condition $Ra_1 = 0$ is satisfied even when it is not required to be so by symmetry alone.

4.2. *Transitions when* $S_0 \neq 0$

Perhaps the simplest type of transition between two three-dimensional space groups is that studied by Lennie *et al.* (1988*a*) using a three-dimensional time-dependent numerical scheme with an infinite Prandtl number and fixed heat flux on both boundaries. This transition is not easy to understand without space groups, and in fact motivated the present investigation. The transition is discussed in detail in Lennie *et al.* and is illustrated in figure 15. It involves the transition from space group $I4/m'mm$ for T_0 and S_0 to $P4/m'mm$ for T_1 and S_1. This transition is particularly simple, because it involves only the loss of the centring symmetry element $t_4 = (L/\sqrt{2}, L/\sqrt{2}, 1)$ in $I4/m'mm$, where L is defined in figure 15. Since the conventional unit cell for this space group is not primitive, whereas that for the perturbation is, the conventional unit cell does not change. The invariant subgroup of $\hat{z}\times T_0$ and $\hat{z}\times S_0$ is $P4/mmm$ and the factor group is isomorphic with Z_2, its cosets are $\{E(P4/mmm), t_4(P4/mmm)\}$ and the transition is of type k:

$$I4/mmm \xrightarrow{k} P4/mmm. \tag{4.2}$$

This transition is important because it is a simple illustration of an active symmetry element involving all three space dimensions that leads to a doubling of the size (in

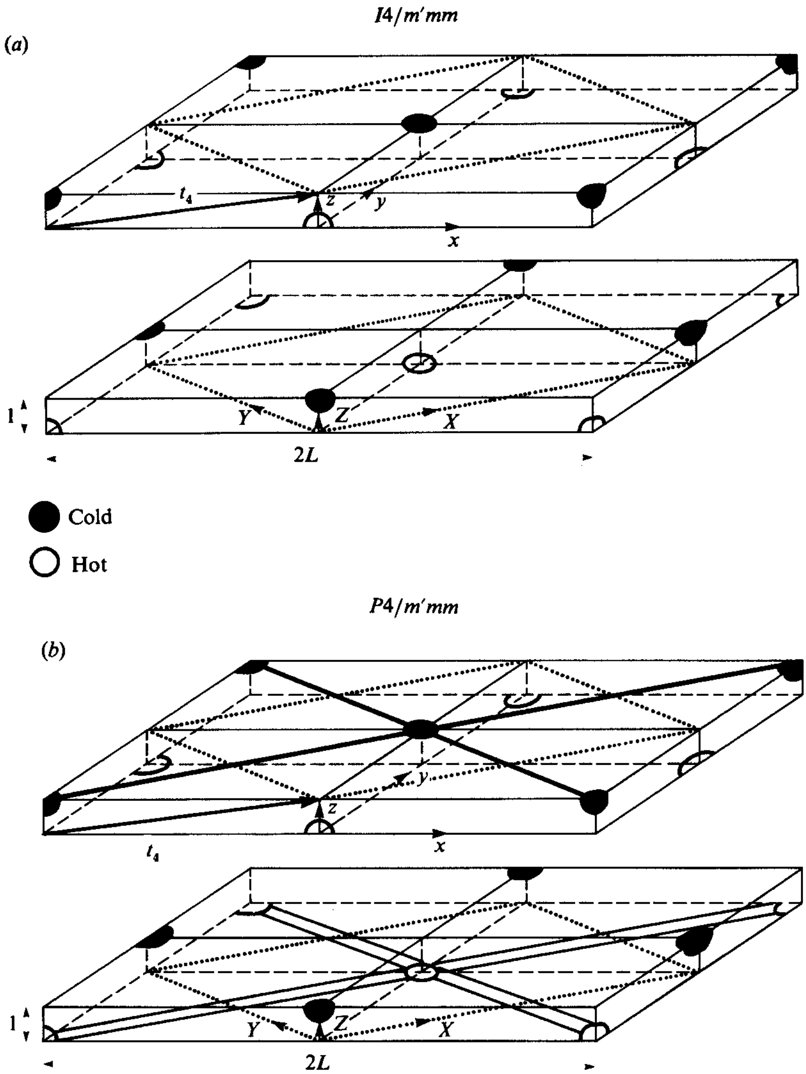

FIGURE 15. A diagrammatic illustration of the circulation determined by Lennie *et al.* (1988*a*) for the Boussinesq problem with no internal heating and fixed heat flux on $z = 0, 1$. The solid dots show sinking regions and open dots rising ones. The numerical experiment was carried out in a region one-eighth the size of that illustrated in (*a*), and had as symmetry group $I4/m'mm$, with a unit cell shown dotted. The centring translation shown as t_4 was lost in a k transition to produce $P4/m'mm$ in (*b*). x, y and z show the coordinate system used for the numerical solution, and X, Y and Z are the conventional crystallographic axes. The unit cell has been separated into two halves on either side of the m'_z plane.

this case the volume) of the unit cell. The steady-state symmetry equations (2.57) and (2.61) are

$$S_0 \times T_1 + T_0 \times S_1 + (\hat{x} \wedge \hat{y}) \times \hat{z} \times T_0 \times \psi_1 = \hat{z} \times T_1, \tag{4.3}$$

$$\frac{1}{\sigma}[(\hat{x} \wedge \hat{y}) \times \hat{z} \times S_0 \times \psi_1 + S_0 \times S_1] = (\hat{x} \wedge \hat{y}) \times \psi_1 + \hat{z} \times S_1 + Ra_0\, \hat{z} \times T_1 + Ra_1\, \hat{z} \times T_0. \tag{4.4}$$

Since Lennie *et al.* (1988*a*) only considered infinite Prandtl numbers, the left-hand side of (4.4) is zero and $\psi_1 = 0$. Then $\hat{z} \times T_0$, $\hat{z} \times S_0 = \Gamma_1$, and $\hat{z} \times T_1$, $\hat{z} \times S_1 = \Gamma_2$. The transition is a pitchfork bifurcation. Since the space group of T_1 and S_1 contains the black-and-white generator t_4' it is a type IV Shubnikov space group (Bradley & Cracknell 1972, figure 7.4).

The best known transitions to circulations with three-dimensional symmetry elements are those that affect rolls. Many have been described by Busse and his colleagues (Busse 1967*b*; Busse & Whitehead 1971; Busse & Clever 1979), and Bolton, Busse & Clever (1986) provide a useful review of earlier work and list many of the instabilities. All except one of these bifurcations are to three-dimensional circulations with black-and-white symmetry elements, and are not easily represented in two-dimensional diagrams. The problem is even greater for the four-dimensional black-and-white space groups needed in discussions of the time-dependent circulations that Bolton *et al.* (1986) describe. However the principles involved in analysing the nature of the bifurcations are the same as those used in the simpler problems already discussed. The presence of elements representing symmetry operations in three and four dimensions does not complicate the structure of the factor groups.

Bolton *et al.* (1986) studied the stability of rolls to a disturbance of the form

$$G(y,z)\,\mathrm{e}^{\mathrm{i}(dy+bx)+st}. \tag{4.5}$$

If s is real, steady circulations may exist when s is small and positive. Oscillatory flows may occur when $s = \mathrm{i}\omega$ where ω is real, and are stable when their amplitude is sufficiently small. Bolton *et al.* classify the disturbances in terms of their symmetry. All the circulations they discuss are invariant under m_z' if the origin is taken to be at the base of the layer, instead of using their origin, which is equidistant from the boundaries. Their 'y-symmetry' corresponds to the element m_y. As their equation (2.3) shows, in the presence of m_y their 'R-symmetry' is not a reflection operation but a black-and-white rotation $C_{2x}^{*\prime}$ through $(0, t_2/2, t_3/2)$, where the dimensions of the unit cell are $2t_1$, $2t_2$ and $2t_3$ in the x-, y- and z-directions. This symmetry operation produces a displacement in the z-direction, and therefore the relevant symmetry groups are three-dimensional space groups in all cases except the Ekhaus instability. It alone can be described by a plane group, but in the (y,z)- not the (x,y)-plane. If $C_{2x}^{*\prime}$ is expressed in terms of conventional symmetry elements, the space groups involved can be found using IT (1983):

$$C_{2x}^{*\prime} = t_2\,t_3\,C_{2x}' = t_2\,t_3\,m_y\,m_z'. \tag{4.6}$$

Since all the circulations are invariant under m_z', the corresponding generator is $t_2\,t_3\,m_y$, or $t_2\,t_3$ when m_y is also an element of the space group. In the case of the zig-zag and E-oscillatory instabilities m_y' is an element of the group. However the space group of the perturbation only agrees with that observed if $t_2\,t_3$, instead of $t_2\,t_3'$, is taken to be a symmetry element (F. H. Busse, personal communication 1987). $t_2\,t_3$ is therefore given in table 1, though this symmetry element is not consistent with equation (2.3) of Bolton *et al.* (1986).

The symmetry elements in the x- and t-directions are easily obtained from (4.5). The steady solution changes sign when translated through $t_1 = (\pi/b, 0, 0)$ and therefore has a symmetry element t_1'. The oscillatory circulations have in addition an element t_t', where $t_t = (0,0,0,\pi/\omega)$.

Table 1 shows the symmetry elements of those circulations described by Bolton *et al.* (1986), together with that of the basic rolls. Many of the symmetry elements of BO2 and BE1 are visible in Bolton *et al.*'s figures 2, 4 and 7. The temperature

	'y-symmetry'	'R-symmetry'	Some of the other elements
2D rolls	m_y	$t_2 t_3$	$m_x, m'_z, t_1, t_t, t_1 t_t, t_1 t_2 t_3$
	Steady, T_1		
Cross-roll and knot	m_y	$t_2 t'_3$	$m_x, m'_z, t'_1, t_1 t_2 t_3$
Zigzag	m'_y	$t_2 t_3$	$m_x, t_1 m_y, m'_z, t'_1$
Eckhaus	—	—	$m_x, t_2 t_3 m_y, m'_z, t_1$
	Oscillatory, T_1		
BO2	m_y	$t_2 t'_3$	$m_x, m'_z, t'_1, t'_t, t_1 t_t, t_1 t_2 t_3$
BE1	m_y	$t_2 t_3$	$m_x, m'_z, t'_1, t'_t, t_1 t_t$
E-Oscillatory	m'_y	$t_2 t_3$	$m_x, t_1 m_y, m'_z, t'_1, t'_t, t_1 t_t$

TABLE 1. Symmetry elements of S and T (modified from Bolton *et al.* 1986). All space groups also contain the generators t_2^2 and t_3^2.

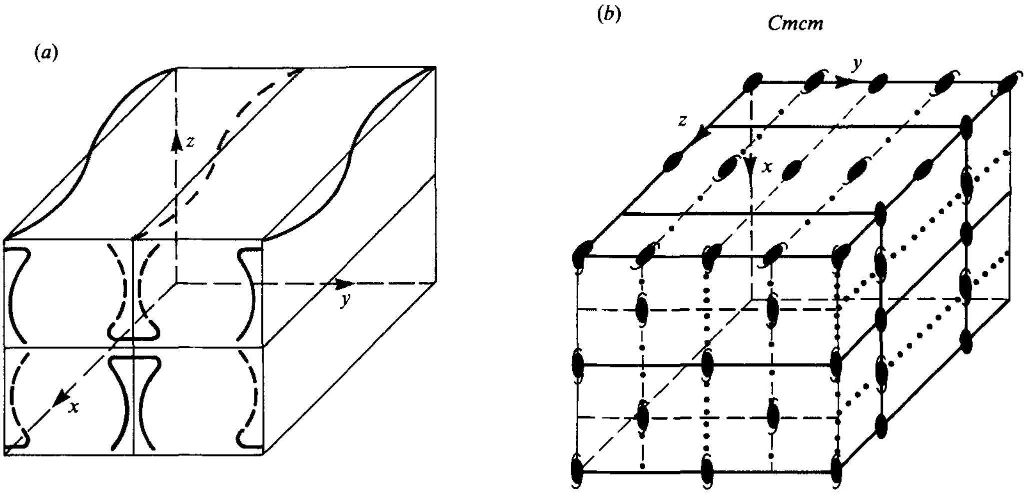

FIGURE 16. A sketch of the isotherms (a) for the zigzag instability with the conventional fluid-dynamical choice of axes. The curves in the (x, y)-plane show the planform of the instability, with the cold sheet shown dashed. Two layers are illustrated to show the presence of m'_z. With the exception of centres of symmetry, the symmetry elements of *Cmcm* are shown in (b) using the conventional choice of crystallographic axes. They correspond to *Amam* when the axes in (a) are used instead.

structure $T_0 + T_1$ and some of the symmetry elements of the zigzag and E-oscillatory instabilities is illustrated in figure 16, where the rolls are displaced by a constant distance on planes with constant x (Busse & Whitehead 1971; Busse 1972).

It is straightforward to use table 1 to determine the generators of the invariant subgroups and the factor groups involved in the bifurcations to steady flows because all elements P and Q satisfy $PQ = QP$. The generators of the invariant subgroups are therefore all the symmetry elements common to the rolls and the perturbation. In the case of the cross-roll and knot instabilities these elements are

$$m_x, \quad m_y, \quad m'_z, \quad t_1^2, \quad t_2^2, \quad t_3^2, \quad t_1 t_2 t_3, \tag{4.7}$$

and the space group of T_1 and S_1 is $Immm'$ ($Im'mm$). The space group of $\hat{z} \times T_1$, $\hat{z} \times S_1$ and $(\hat{x} \wedge \hat{y}) \times \psi_1$ is therefore $Immm$, the active element is t_1, and the factor group is

$$Z_2 = \{E(Immm), \quad t_1(Immm)\}, \tag{4.8}$$

and the instability is a pitchfork bifurcation. The space group for the zigzag instability differs from that of the cross-roll because it contains the elements $t_2 t_3$ and $t_1 m_y$ instead of $t_2 t'_3$ and m_y, and therefore $Immm$ is replaced by the non-symmorphic group $Amam(Cmcm)$ in (4.8).

All the symmetry elements for the Eckhaus instability are contained in the space group for the rolls. It is therefore not possible to show that this transition is a pitchfork bifurcation from symmetry alone.

Unlike the orthorhombic symmetry of all the space groups discussed above, the skewed varicose instability (Busse & Clever 1979; Busse & Bolton 1984) has a monoclinic space group, though it has not been possible to determine all the elements of the space group of T_1 from the published information. But, since the expression (4.5) for T_1 has a symmetry element t'_t, whereas that for T_0 has an element t_t, the transition must be a pitchfork bifurcation.

4.3. *General remarks about space-group transitions in constant-viscosity circulations*

In contrast to the great variety of planform transitions observed by White (1981, 1988), rather few transitions to steady flows in constant-viscosity fluids have yet been described. Most of those that have been studied are k transitions, and all have been to the maximal invariant subgroup. There is as yet no indication of how the critical Rayleigh number varies with the order of the factor group when the transition is to a non-maximal k subgroup, or when successive k transitions occur. Except for the Eckhaus instability, all the transitions yet studied in detail involve perturbations with the symmetry of type IV Shubnikov black-and-white space groups, and have Z_2 as a factor group.

The difference in the behaviour between the constant- and temperature-dependent-viscosity circulations is striking. It results from the stability of rolls at marginal stability when the viscosity is constant. The translational symmetry element along the roll axes must be lost to produce a three-dimensional circulation, and in most cases the point-group symmetry elements are retained. Little work has yet been carried out on transitions between the resulting three-dimensional circulations. Only in the case of the skewed varicose instability is the symmetry element m_y of the rolls also lost. In contrast rolls are not a stable planform in variable-viscosity circulations when the variation of viscosity is large, and therefore transitions between plane groups are easily observed at moderate Rayleigh numbers. Similar transitions may be difficult to study in constant-viscosity circulations because of Hopf bifurcations to time-dependent flows.

The principal complication in using group theory to discuss marginal stability when the viscosity is constant is that the linearized equations are self-adjoint, and therefore all solutions will bifurcate from the stationary state with $Ra_1 = 0$, even when not required to do so by their symmetry. This complication illustrates a general result: $Ra_1 = 0$ on symmetry grounds alone is a sufficient but not a necessary condition for a pitchfork bifurcation.

5. Symmetry elements involving time

5.1. *Hopf bifurcations and loss of m_t*

Unlike the reflection symmetry in the space domain, no solution that shows time dependence can contain either m_t or m'_t as a symmetry operation. This result is well known, and is obvious from the form of the symmetry equations, which involve both $\hat{t} \times T_1$ and T_1. Oscillatory solutions are a common feature of convective circulations, and may involve temperature perturbations that are symmetric under m_t, when the origin is appropriately chosen. It is then not obvious how this result is compatible with the structure of the equations. The purpose of this section is to examine a variety of transitions to time-dependent flow and show that in all these cases the Hopf bifurcation involves the factor group D_2, and that the full solution is not invariant under m_t, though the temperature may be if another variable is not. As in the case of marginal stability, the symmetry group of the basic steady-state flow is a Lie group. Therefore the discussion below is incomplete, though the results are correct. A full treatment will be published elsewhere (D. McKenzie, in preparation).

A well-known Hopf bifurcation that has been studied extensively occurs in double-diffusive convection. The mechanism of the oscillation was outlined by Stern (1960) and Veronis (1965), who emphasized that the disturbance to the temperature field was not in phase with that to the solute. This result is easily obtained from (2.23) and (2.24). Elimination of a_1 and substitution of

$$a_3 = a_3^0 \exp(\mathrm{i}\omega t), \quad a_5 = a_5^0 \exp(\mathrm{i}\omega t) \tag{5.1}$$

gives
$$a_5 = \left(\frac{\tau-\Omega^2}{\tau^2-\omega^2} - \frac{\mathrm{i}\Omega(1-\tau)}{\tau^2-\Omega^2}\right) a_3, \tag{5.2}$$

where
$$\Omega = \omega/\pi^2(\alpha^2+1). \tag{5.3}$$

If a_3^0 is real, the temperature disturbance is symmetric about $t = 0$. However a_5^0 is then only real if $\tau = 1$, or $\kappa_s = \kappa_T$. Oscillations cannot then occur. The same result can be proved more generally using factor groups. The stationary state that is being perturbed has linear temperature and salinity gradients and $S_0 = 0$. T_1, S_1 and s_1 have the plane-group symmetry $a2'mm'(c2'm'm)$ shown in figure 13. If the period of the oscillation is $2t_t$ the appropriate factor group is

$$\left.\begin{array}{ccccc}
D_2 & E(a2'mm') & m_t & t_t & m_t t_t \\
\Gamma_1 & 1 & 1 & 1 & 1 \\
\Gamma_2 & 1 & -1 & 1 & -1 \\
\Gamma_3 & 1 & 1 & -1 & -1 \\
\Gamma_4 & 1 & -1 & -1 & 1
\end{array}\right\}. \tag{5.4}$$

Since the flow is two-dimensional $\psi = 0$. Also

$$\hat{z} = T_0 = s_0 = \Gamma_1, \quad \hat{t} = \Gamma_2. \tag{5.5}$$

Thus (2.57), (2.59) and (2.61) become

$$\Gamma_2 \times T_1 + S_1 = T_1, \tag{5.6}$$

$$\Gamma_2 \times s_1 + S_1 = \tau s_1, \tag{5.7}$$

and
$$\frac{1}{\sigma}\Gamma_2 \times S_1 = S_1 + Ra_0 T_1 + Ra_1 \Gamma_1 + R_{s0} s_1 + R_{s1} \Gamma_1. \tag{5.8}$$

The condition that $Ra_1 = R_{s1} = 0$ requires S_1, T_1 and s_1 to contain no component that transforms as does Γ_2. This condition is satisfied provided the perturbation contains no components with period t_t. Then all can be written in terms of Γ_3 and Γ_4. Furthermore, substitution of S_1 from (5.6) into (5.8) gives

$$\frac{1}{\sigma}(\Gamma_2 \times T_1 + T_1) = \Gamma_2 \times T_1 + T_1 + Ra_1 \Gamma_1 + Ra_0 T_1 + R_{s0} s_1 + R_{s1} \Gamma_1. \tag{5.9}$$

If $T_1 = \Gamma_3$ then s_1 must contain both Γ_3 and Γ_4, $Ra_1 = R_{s1} = 0$ and the transition is a Hopf bifurcation. The same is true when $T_1 = \Gamma_3 + \Gamma_4$.

It is of some interest to use the structure of the factor group to understand how the oscillation arises. The essential feature is that the time operator $\hat{t}$ is active, but transforms differently from either s_1 or T_1. Because $\Gamma_2 \times \Gamma_3 = \Gamma_4$ and $\Gamma_2 \times \Gamma_4 = \Gamma_3$, the active operator generates Γ_4 from Γ_3 and Γ_3 from Γ_4. It is this behaviour that produces the oscillation. The active operator does not, however, generate Γ_1 and therefore $Ra_1 = 0$, the transition is a Hopf bifurcation, and the amplitudes of both Γ_3 and Γ_4 are proportional to $(Ra - Ra_0)^{\frac{1}{2}}$. This type of behaviour is not possible if the factor group is Z_2 or D_3. The smallest group for which it can occur is D_2, and such a group arises when two symmetry elements P and Q are involved in the transition, with $PQ = QP$, $P^2 = Q^2 = E$. This condition is satisfied by the cosets $m_t\{a2'mm'\}$ and $t_t\{a2'mm'\}$ in the previous example.

Two Hopf bifurcations have been studied in two-dimensional circulations which can both be discussed using three-dimensional space groups. In low-Prandtl-number fluids the even or E-oscillatory instability generates wiggly rolls (Willis & Deardorff 1970; Busse 1972; Bolton *et al.* 1986). As table 1 shows, at any instant the temperature distribution has the same space group as the zigzag instability in figure 16. Busse (1972) compares the instability to waves propagating along a rope, and also shows that the instability is still present when $b = 0$ in (4.5). Bolton *et al.* (1986) show that the growth rate of this instability is strongly dependent on the Prandtl number, which suggests that the nonlinear terms in the momentum equation are important. In two dimensions $\psi = 0$ and the symmetry equations (2.57) and (2.61) become

$$\hat{t} \times \hat{z} \times T_1 + S_0 \times T_1 + T_0 \times S_1 = \hat{z} \times T_1 \tag{5.10}$$

and
$$\frac{1}{\sigma}(\hat{t} \times \hat{z} \times S_1 + S_0 \times S_1) = \hat{z} \times S_1 + Ra_0\, \hat{z} \times T_1 + Ra_1\, \hat{z} \times T_0 + P_1. \tag{5.11}$$

When $b = 0$ table 1 shows that the elements of the space group of T_1 include

$$t_t m_y, \quad m'_z, \quad t'_t, \quad t_t t_2 t_3, \quad t_t^2, \quad t_2^2, \quad t_3^2. \tag{5.12}$$

If $\hat{t}$ is taken to be in the x-direction, the space group of S_1 and T_1 is $I2'am'(Im'a2')$ and that for $\hat{z} \times S_1$ and $\hat{z} \times T_1$ is therefore $I2am(Ima2)$. The factor group can be written

$$D_2 = \{E(I2am), \quad m_t(I2am), \quad t_t(I2am), \quad t_t m_t(I2am)\}. \tag{5.13}$$

The oscillation arises in exactly the same way as in the previous example, with $\hat{t} = \Gamma_2$; $\hat{z} \times T_1$, $\hat{z} \times S_1 = \Gamma_3 + \Gamma_4$. Since $S_0 \times T_1$, $T_0 \times S_1$ and $S_0 \times S_1$ are all invariant under the same transformations, it is not possible to decide which of these nonlinear terms maintains the oscillation from symmetry alone. But the strong dependence on Prandtl number suggests that $S_0 \times S_1$ is responsible.

A rather different two-dimensional Hopf bifurcation arises when the Prandtl number is infinite, when (5.11) becomes

$$0 = \hat{z} \times S_1 + Ra_0\, \hat{z} \times T_1 + Ra_1\, \hat{z} \times T_0 + P_1 \tag{5.14}$$

and corresponds to a circulation like BE1 or BO2 in table 1 with $b = 0$. The elements include

$$m_y, \quad m'_z, \quad t'_t, \quad t_t t_2 t_3, \quad t_t^2, \quad t_2^2, \quad t_3^2. \tag{5.15}$$

If $\hat{t}$ is again taken to be in the x-direction the space group for T_1 and S_1 is $I2'mm'$, ($Im'm2'$). The invariant space group required is therefore $I2mm(Imm2)$ and the factor group is

$$D_2 = \{E(I2mm), \quad m_t(I2mm), \quad t_t(I2mm), \quad t_t m_t(I2mm)\}. \tag{5.16}$$

This Hopf instability was investigated in a fluid with finite Prandtl number by Moore & Weiss (1973). The time-dependent behaviour studied by Curry *et al.* (1984) and the Hopf bifurcation of Lennie *et al.* (1988*b*) arise in the same way. The physical mechanism that produces the instability is the advection of a number of hot and cold blobs by the basic flow. As the active elements are m_t and t_t, the number of blobs does not affect the nature of the transition. This remark is in agreement with the behaviour reported by Curry *et al.* (1984).

Three sketches of the geometry of T_1 are illustrated in figure 17 (*a–c*), corresponding to the advection of one, two and three pairs of hot and cold blobs around the cell. Moore & Weiss (1973, figure 4*a*) illustrate the time dependence of the vertical, w, and horizontal, u, velocities. These show a small distortion from harmonic motion that is most noticeable in u. This distortion removes the symmetry element m_t that would otherwise be present in one curve. Since such an element cannot be present in the space group of the perturbation as a whole, its loss can have no effect on the nature of the transition.

The three Hopf bifurcations in table 1 are similar to those discussed above except that the invariant subgroups are four-dimensional and black and white. Though the tables of Brown *et al.* (1978) list all four-dimensional space groups and their generators, they are not easy to use. The generators are written in terms of the primitive basis vectors, and, unlike the conventions used in IT (1983), the labels used for the space groups give no indication of the symmetry elements they contain. The generators in table 1 show that all three space groups are hyperorthorhombic, with two reflection elements. Two contain a third reflection, whereas the E-oscillatory instability has a glide plane and is therefore non-symmorphic. The groups involve two different types of centring, that of BO2 being $G(1,4)$ if the time axis is taken to be x_4, whereas those of BE1 and the E-oscillatory circulation belong to $D(1,4)(2,3)$. Further progress requires the generators of all groups with these lattices to be converted to the basis used in table 1 by using the Z matrices listed on pp. 271–2 of Brown *et al.* (1978). The Z-class (Brown *et al.* 1978) of BE1 and the E-oscillatory instabilities is 06/01/07, and that of BO2 is 06/01/10. The generators given as A, B and C for these groups (Brown *et al.* 1978, p. 97) are $m_y, m_t, m_y m_z$ for 06/01/10 and $m_x, m_t, m_x m_y$ for 06/01/07. For $D(1,4)(2,3)$ centring the vectors $(1,1,0,0)/2$ and $(0,0,1,1)/2$ correspond to t_t and t_3 respectively. The space group of $\hat{z}\times S_1$, $\hat{z}\times T_1$ and $(\hat{x}\wedge\hat{y})\times\psi_1$ for the E-oscillation can then be identified as the non-symmorphic group 06/01/07/003, and the symmorphic groups 06/01/10/001 and 06/01/07/001 for BO2 and BE1 respectively. If any of the three invariant subgroups is denoted by K, then the factor group for all of these instabilities is

$$D_2 = \{E(K), \quad m_t(K), \quad t_t(K), \quad t_t m_t(K)\} \tag{5.17}$$

and the oscillations arise in the same way as before. Therefore all six Hopf bifurcations involve the same factor group D_2, and differ only in their invariant subgroups.

A general difficulty in analysing the Hopf transitions discussed above is the absence of detailed contour plots of the structure of the oscillatory flow. In the case of double-diffusive convection this problem does not arise because the perturbation

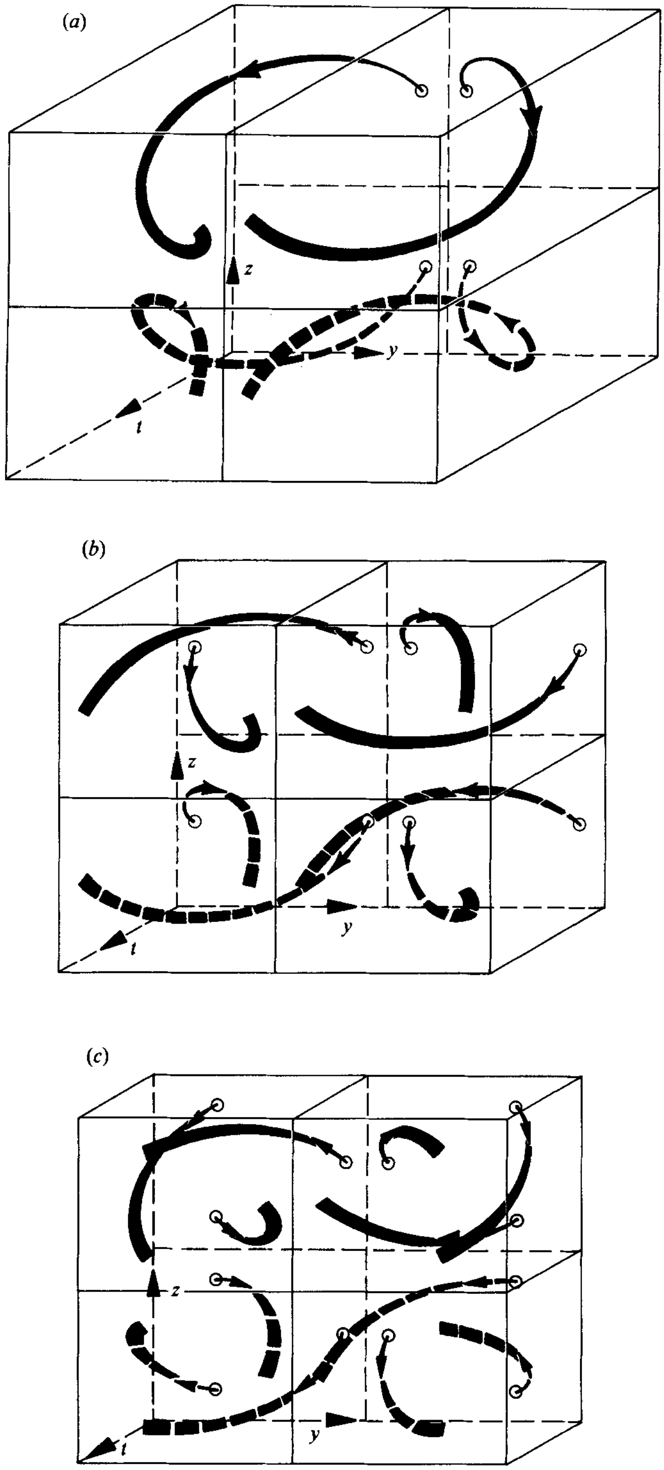

FIGURE 17. Sketch to illustrate the symmetry in (t, y, z)-space of various time-dependent solutions, showing the path of hot blobs as heavy solid lines and cold blobs as heavy dashed lines. (*a*) One hot blob. (*b*) Curry *et al.*'s (1984) oscillation with two hot blobs. (*c*) Moore & Weiss' (1973) circulation with three hot blobs. None of the solutions possess m_t as a symmetry element.

equations (2.23) and (2.24) are available. What is needed is plots of the spatial and temporal behaviour of the flow and tables like that of Bolton *et al.* (1986). The plots should show perturbations such as T_1, and not either T_0+T_1, or spatially averaged functions such as the Nusselt number, or nonlinear functions such as the amplitude or the kinetic energy.

5.2. *k transitions involving* t_t

Once a Hopf bifurcation has occurred to produce a periodic solution the symmetry group can change through t and k transitions. The active elements may involve one, two, three or four dimensions. Most work has been carried out on the Lorenz equations (see Sparrow 1982), where the bifurcations occur as the Rayleigh number is reduced for the usual choice of parameters. Period doubling takes place, with a transition to the maximal subgroup of the infinite line group corresponding to displacements in time by one period t_t. The active element in this transition is t_t, the differential operator d_t is passive, the factor group is $Z_2 = \{E(\not{p}), t_t(\not{p})\}$ and each transition is a pitchfork bifurcation. The reason why this behaviour appears to be different from the spatial period doubling illustrated in figure 15 is that the only symmetry element present in the solutions to the Lorenz equations is a translation in time, whereas three-dimensional convective circulations generally contain a number of point symmetry elements as well. The factor groups involved are, however, the same, and the period doubling changes the periodicity in time rather than in space.

6. Discussion

The detailed discussion of the symmetry of many of the transitions observed in convecting systems and listed in table 2 has obscured the essential simplicity of the results obtained. Transitions to steady-state circulations involve loss of either a point-group symmetry element (t transitions) or a lattice symmetry element (k transitions). There are only a small finite number of point-group elements present, but an infinite number of translation elements exist in all line, plane and space groups. Only one point-group element, m, exists in line groups, but several are present in plane and space groups. Most transitions to steady circulations that have been observed occur through pitchfork bifurcations. This result is easy to demonstrate when the ordinary symmetry group of the perturbation is an invariant subgroup of the unperturbed symmetry group, with an additional black and white element. When this is not the case the factor group is not isomorphic with Z_2, but with D_3 or D_5. The discussion of transitions from planforms with a symmetry element that is a continuous variable to one with a discrete rotational or translational element is incomplete, and a full treatment will be published elsewhere (D. McKenzie, in preparation).

In all the examples discussed in §5, the transition from steady to oscillatory flow involves the loss of two symmetry elements to produce a D_2 factor group. The oscillation results from the $\hat{t}$ operator having a different transformational behaviour from the temperature and stream function, which are combinations of two other representations Γ_3 and Γ_4, but not of the trivial representation Γ_1. This behaviour produces a Hopf bifurcation. The convection system then in general has four-dimensional symmetry elements and must be represented by a four-dimensional black-and-white space group, though some of the examples discussed can be

Section	Instability		Factor group	Figure No.
	Squares and rectangles			
3.1	Mosaic (White 1988, figure 13)	$p4mm \xrightarrow{k} p4mm$	Z_2	2
3.1	Cross rolls (White 1988, figure 23)*	$p_{01}2mm \xrightarrow{k} p2mm$	Z_2	3
3.2	Cell stretching (White 1981)	$p4mm \xrightarrow[C_{4z}]{t} p2mm$	Z_2	4
3.2	Cell splitting (White 1981)	$p2mm \xrightarrow{k} p2mm$	Z_2	5
3.2	Cell fusion (White 1988, figure 13)	$p2mm \xrightarrow{k} c2mm$	Z_2	6
3.2	Lip (White 1981)	$p2mm \xrightarrow[C_{2z}]{t} pm$	Z_2	7
3.3	From hexagons (White 1981)	$c2mm \xrightarrow{k} p2mg$	Z_2	9
3.3	From hexagons (White 1981)	$p2mm \xrightarrow{k} p2mg$	Z_2	—
3.3	Cell splitting (White 1988, figure 18)	$c2mm \xrightarrow{k} c2mm$	D_5	10
	Hexagons			
3.3	Hexagons to rolls (Richter 1978; White 1981)	$c6mm \xrightarrow[C_{3z}]{t} c2mm$	D_3	8
3.3	Mosaic with hexagonal planform (White 1988, figure 16)	$c6mm \xrightarrow{k} c6mm$	D_3	11
	Space groups			
4.1	Marginal instability to rolls*	$P_{001}\,mmm \xrightarrow{k} A_{011}\,mmm$	Z_2	13
4.1	Marginal instability to hexagons*	$P_{001}\,mmm \xrightarrow{k} C6/mmm$	Z_2	14
4.2	Cross rolls (Busse 1967*b*)*	$A_{011}\,mmm \xrightarrow{k} Immm$	Z_2	—
4.2	Squares (Lennie *et al.* 1988*a*)	$I4/mmm \xrightarrow{k} P4/mmm$	Z_2	15
4.2	Zigzag (Busse 1967*b*)*	$A_{011}\,mmm \xrightarrow{k} Amam(Cmcm)$	Z_2	16

TABLE 2. Pitchfork bifurcations. The plane and space groups are those of $\hat{z}\times S$, $\hat{z}\times T$ and $(\hat{x}\wedge\hat{y})\times\psi$. Transitions marked with an asterisk involve representations of Lie groups, and are discussed in detail elsewhere (D. McKenzie, in preparation).

represented using only three dimensions. Presumably t and k transitions will be discovered in four-dimensional systems. By analogy with the three-dimensional behaviour, most transitions are likely to be pitchfork bifurcations.

A striking feature of all the factor groups is that they are either Z_2 or dihedral groups, even when the transition is a Hopf bifurcation. This behaviour arises because the factor groups all have Z_2 as an invariant subgroup. Direct products of Z_2 with Z_1 and Z_2 produce Z_2 and D_2, semidirect products with Z_3 and Z_5 generate D_3 and D_5. The significance of this behaviour is at present obscure, though it would not be surprising if it was the expression of some more general principal.

Another general result of considerable interest is that the symmetry group of all convective circulations that have yet been studied contains reflection as a symmetry element, and therefore all bifurcations yet observed conserve parity. Since all the bifurcations discussed above arise through a spontaneous loss of a symmetry element, there is no obvious reason why m should be retained when other elements are lost. But parity is commonly conserved in physical systems, and it is therefore not surprising that it is conserved in many convective transitions.

The only time-dependent system that has yet been studied in detail is that governed by the Lorenz equations. Since these consist of nonlinear ordinary differential equations, the solutions are periodic in time only. Since the time-dependent solutions do not have m_t as a symmetry element, only k transitions are possible. These lead to period-doubling cascades, with a Z_2 factor group, and the critical Rayleigh numbers of the successive transitions become more and more similar. When one more dimension is added to the system, the transitions occur between plane groups and the behaviour is more diverse. Both t and k transitions occur in two (§3) and three (§4) dimensions. In the case of plane groups there is also a suggestion that the differences between the critical Rayleigh numbers of successive k transitions become smaller. If this is indeed the case then period-doubling cascades should occur. But a more detailed investigation of this question is required. At present no k transitions involving translations in space and time have been reported, though suitable symmetry elements are present in the circulations studied by Bolton *et al.* (1986).

In the light of these results, period-doubling cascades are to be expected in two-, three- and four-dimensional convective transitions. The consequences of their existence may already have been observed: all laboratory experiments that have been carried out at large Rayleigh numbers without imposing an initial planform with a plane-group symmetry have led to spatially aperiodic flows. By analogy with the Lorenz system, the simplest process that could lead to aperiodic circulation in space and time is period doubling with an active element that is a space–time displacement, like that observed by Bolton *et al.* (1986). However, it is unlikely that only k transitions will be involved. Most fluid dynamicists would describe a system that is aperiodic in space and time as turbulent. If convective systems do indeed become turbulent by cascades of pitchfork bifurcations principally involving k transitions in space and time, the process is rather different from Landau's (see Landau & Lifshitz 1959, §27) and Hopf's (1948) suggestions. They argued that successive Hopf bifurcations occurred, producing periodic solutions with more than one modulation frequency. Ruelle & Takens (1971 *a*, *b*) also have suggested that Hopf bifurcations of this type lead to turbulence (see also Arnol'd 1983, p. 278). Though such transitions are well known in solid materials (see for instance Heine & McConnell 1984), commensurate transitions are also common in crystallography and have received less attention in fluid mechanics. As this discussion shows, they also can lead to a behaviour that could be described as turbulent.

As the examples discussed above illustrate, a great variety of pitchfork bifurcations occurs in convective systems. Several of these do not satisfy Landau's conditions for a second-order crystallographic transition. The fluid-mechanical constraints are therefore weaker than the crystallographic, a difference that may also encourage pitchfork bifurcations. To discover whether period-doubling cascades do in fact lead to 'turbulent' behaviour it is necessary to determine the critical Rayleigh numbers for the first few steps in a period-doubling cascade. Such a study is probably most easily carried out numerically. Transitions other than the one of interest can then be suppressed by forcing the solution to have the desired symmetry. The relevant

Fourier expansions for this purpose for all 230 space groups are listed in IT (1952).

Though the examples in the previous sections were all taken from convection in a uniform layer of fluid in the absence of rotation, the methods are quite general and can be applied to any systems of differential equations. Because rather few general methods are available for the study of nonlinear partial differential equations, the methods developed here are likely to be of most use in such problems.

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Appendix

Projection operators can be used to obtain functions that transform in the same way as does a particular irreducible representation. When the representation is one-dimensional the procedure is obvious: all the elements of the group are applied to the function in turn and the resulting expressions combined with the correct signs for the relevant representation. When, however, the irreducible representations are not one-dimensional the procedure is less obvious, and projection operators ρ must be used, where

$$\rho_{ij}^{m} = \frac{d_m}{n} \sum_{n} \Gamma_{ij}^{n} g_n \tag{A 1}$$

and i and j refer to the row and the column of the representation Γ_m of dimension d_m, n is the order of the group and Γ_{ij}^{n} is the ij element of the matrix representing the group element g_n. Knowledge of the character table is not sufficient to evaluate (A 1): it is necessary to know the complete representations of every element in the group. Di Bartolo (1968) provides a worked example of the use of (A 1), and the development below follows the same procedure, which is applied to D_5 to describe the transition in figure 10. This process also illustrates the use of the Fourier expansions obtained from IT (1952).

The first step is to obtain the representations of all elements of the group. The two one-dimensional representations are given in (3.27), and the projection operators can be written down by inspection:

$$\rho_{11}^{1} = \tfrac{1}{10}(E+t_1+t_1^4+t_1^2+t_1^3+C_{2z}+t_1 C_{2z}+t_1^2 C_{2z}+t_1^3 C_{2z}+t_1^4 C_{2z}), \tag{A 2}$$

$$\rho_{11}^{2} = \tfrac{1}{10}(E+t_1+t_1^4+t_1^2+t_1^3-C_{2z}-t_1 C_{2z}-t_1^2 C_{2z}-t_1^3 C_{2z}-t_1^4 C_{2z}). \tag{A 3}$$

To obtain those for Γ_3 and Γ_4 the representations must first be written out. Those for Γ_3 are

$$\left.\begin{aligned}
E &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad t_1 = \begin{pmatrix} c_1 & s_1 \\ -s_1 & c_1 \end{pmatrix}, \quad t_1^4 = \begin{pmatrix} c_1 & -s_1 \\ s_1 & c_1 \end{pmatrix}; \\
t_1^2 &= \begin{pmatrix} c_2 & s_2 \\ -s_2 & c_2 \end{pmatrix}, \quad t_1^3 = \begin{pmatrix} c_2 & -s_2 \\ s_2 & c_2 \end{pmatrix}; \\
C_{2z} &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad t_1 C_{2z} = \begin{pmatrix} c_1 & -s_1 \\ -s_1 & -c_1 \end{pmatrix}, \quad t_1^2 C_{2z} = \begin{pmatrix} c_2 & -s_2 \\ -s_2 & -c_2 \end{pmatrix}, \\
t_1^3 C_{2z} &= \begin{pmatrix} c_2 & s_2 \\ s_2 & -c_2 \end{pmatrix}, \quad t_1^4 C_{2z} = \begin{pmatrix} c_1 & s_1 \\ s_1 & -c_1 \end{pmatrix},
\end{aligned}\right\} \tag{A 4}$$

where $\qquad c_1 = \cos\beta, \quad s_1 = \sin\beta, \quad c_2 = \cos 2\beta, \quad s_2 = \sin 2\beta;$

and for Γ_4 are

$$\left.\begin{aligned}
E &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad t_1 = \begin{pmatrix} c_2 & s_2 \\ -s_2 & c_2 \end{pmatrix}, \quad t_1^4 = \begin{pmatrix} c_2 & -s_2 \\ s_2 & c_2 \end{pmatrix}; \\
t_1^2 &= \begin{pmatrix} c_1 & -s_1 \\ s_1 & c_1 \end{pmatrix}, \quad t_1^3 = \begin{pmatrix} c_1 & s_1 \\ -s_1 & c_1 \end{pmatrix}; \\
C_{2z} &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad t_1 C_{2z} = \begin{pmatrix} c_2 & -s_2 \\ -s_2 & -c_2 \end{pmatrix}, \quad t_1^2 C_{2z} = \begin{pmatrix} c_1 & s_1 \\ s_1 & -c_1 \end{pmatrix}, \\
t_1^3 C_{2z} &= \begin{pmatrix} c_1 & -s_1 \\ -s_1 & -c_1 \end{pmatrix}, \quad t_1^4 C_{2z} = \begin{pmatrix} c_2 & s_2 \\ s_2 & -c_2 \end{pmatrix}.
\end{aligned}\right\} \tag{A 5}$$

The eight projection operators for these two representations can now be written down:

$$\rho_{11}^3 = \tfrac{1}{5}(E + c_1 t_1 + c_1 t_1^4 + c_2 t_1^2 + c_2 t_1^3 + C_{2z} + c_1 t_1 C_{2z} + c_2 t_1^2 C_{2z} + c_2 t_1^3 C_{2z} + c_1 t_1^4 C_{2z}), \tag{A 6}$$

$$\rho_{12}^3 = \tfrac{1}{5}(s_1 t_1 - s_1 t_1^4 + s_2 t_1^2 - s_2 t_1^3 - s_1 t_1 C_{2z} - s_2 t_1^2 C_{2z} + s_2 t_1^3 C_{2z} + s_1 t_1^4 C_{2z}), \tag{A 7}$$

$$\rho_{21}^3 = \tfrac{1}{5}(-s_1 t_1 + s_1 t_1^4 - s_2 t_1^2 + s_2 t_1^3 - s_1 t_1 C_{2z} - s_2 t_1^2 C_{2z} + s_2 t_1^3 C_{2z} + s_1 t_1^4 C_{2z}), \tag{A 8}$$

$$\rho_{22}^3 = \tfrac{1}{5}(E + c_1 t_1 + c_1 t_1^4 + c_2 t_1^2 + c_2 t_1^3 - C_{2z} - c_1 t_1 C_{2z} - c_2 t_1^2 C_{2z} - c_2 t_1^3 C_{2z} - c_1 t_1^4 C_{2z}), \tag{A 9}$$

$$\rho_{11}^4 = \tfrac{1}{5}(E + c_2 t_1 + c_2 t_1^4 + c_1 t_1^2 + c_1 t_1^3 + C_{2z} + c_2 t_1 C_{2z} + c_1 t_1^2 C_{2z} + c_1 t_1^3 C_{2z} + c_2 t_1^4 C_{2z}), \tag{A 10}$$

$$\rho_{12}^4 = \tfrac{1}{5}(s_2 t_1 - s_2 t_1^4 - s_1 t_1^2 + s_1 t_1^3 - s_2 t_1 C_{2z} + s_1 t_1^2 C_{2z} - s_1 t_1^3 C_{2z} + s_2 t_1^4 C_{2z}), \tag{A 11}$$

$$\rho_{21}^4 = \tfrac{1}{5}(-s_2 t_1 + s_2 t_1^4 + s_1 t_1^2 - s_1 t_1^3 - s_2 t_1 C_{2z} + s_1 t_1^2 C_{2z} - s_1 t_1^3 C_{2z} + s_2 t_1^4 C_{2z}), \tag{A 12}$$

$$\rho_{22}^4 = \tfrac{1}{5}(E + c_2 t_1 + c_2 t_1^4 + c_1 t_1^2 + c_1 t_1^3 - C_{2z} - c_2 t_1 C_{2z} - c_1 t_1^2 C_{2z} - c_1 t_1^3 C_{2z} - c_2 t_1^4 C_{2z}). \tag{A 13}$$

Projection operators act on the perturbed temperature in figure 10(*b*), whose symmetry group is *c2mm*. IT (1952, p. 370) gives the general terms in the Fourier expansion of *c2mm* as

$$\left.\begin{aligned}
A &= 8 \cos 2\pi h x^* \cos 2\pi k y^*, \\
B &= 0,
\end{aligned}\right\} \tag{A 14}$$

where $h+k = 2n$ and n is an integer, and the variables marked with an asterisk are scaled so that the unit cell extends from 0 to 1 in the x^*- and y^*-directions. In terms of the variables x and y used in §3.3,

$$x^* = \frac{x}{5t_1}, \quad y^* = \frac{y}{t_2}. \tag{A 15}$$

c2mm can then be written as

$$f(x,y) = \sum_h \sum_k [A(h,k)+B(h,k)], \tag{A 16}$$

where $f(x,y)$ is any function with the symmetry group *c2mm*.

All that now remains is to write out the expressions produced by each of the group operations on A and then to evaluate the projections. Ignoring the numerical factor of eight in (A 14), the group elements give

$$\left.\begin{aligned} EA &= \cos(\beta hX)\,c_k, & t_1 A &= \cos(\beta h(X+1))\,c_k,\\ t_1^4 A &= \cos(\beta h(X-1))\,c_k, & t_1^2 A &= \cos(\beta h(X+2))\,c_k,\\ t_1^3 A &= \cos(\beta h(X-2))\,c_k, & C_{2z} A &= \cos(\beta hX)\,c_k,\\ t_1 C_{2z} A &= \cos(\beta h(X-1))\,c_k, & t_1^2 C_{2z} A &= \cos(\beta h(X-2))\,c_k\\ t_1^3 C_{2z} A &= \cos(\beta h(X+2))\,c_k, & t_1^4 C_{2z} A &= \cos(\beta h(X+1))\,c_k \end{aligned}\right\} \tag{A 17}$$

where $$c_k = \cos 2\pi k \frac{y}{t_2}, \quad X = \frac{x}{t_1}.$$

Evaluation of (A 6)–(A 13) using

$$1+2\cos\beta+2\cos 2\beta = 0 \tag{A 18}$$

gives

$$\left.\begin{aligned} \rho^1_{11} &= D_5(m,k)\cos((10m+5)\beta X)\,c_k, && k \text{ odd}\\ &= D_{10}(m,k)\cos((10m+10)\beta X)\,c_k && k \text{ even}\\ \rho^2_{11} &= 0, &&\\ \rho^3_{11} &= (D_1(m,k)\cos((10m+1)\beta X)+D_9(m,k)\cos((10m+9)\beta X))\,c_k && k \text{ odd}\\ &= (D_4(m,k)\cos((10m+4)\beta X)+D_6(m,k)\cos((10m+6)\beta X))\,c_k, && k \text{ even}\\ \rho^3_{12} &= (-D_1(m,k)\sin((10m+1)\beta X)+D_9(m,k)\sin((10m+9)\beta X))\,c_k && k \text{ odd}\\ &= (D_4(m,k)\sin((10m+4)\beta X)-D_6(m,k)\sin((10m+6)\beta X))\,c_k, && k \text{ even}\\ \rho^3_{21} &= 0, \quad \rho^3_{22} = 0, &&\\ \rho^4_{11} &= (D_3(m,k)\cos((10m+3)\beta X)+D_7(m,k)\cos((10m+7)\beta X))\,c_k && k \text{ odd}\\ &= (D_2(m,k)\cos((10m+2)\beta X)+D_8(m,k)\cos((10m+8)\beta X))\,c_k, && k \text{ even}\\ \rho^4_{12} &= (D_3(m,k)\sin((10m+3)\beta X)-D_7(m,k)\sin((10m+7)\beta X))\,c_k && k \text{ odd}\\ &= (-D_2(m,k)\sin((10m+2)\beta X)+D_8(m,k)\sin((10m+8)\beta X))\,c_k, && k \text{ even}\\ \rho^4_{21} &= 0, \quad \rho^4_{22} = 0 && \end{aligned}\right\} \tag{A 19}$$

where $D_1 \ldots D_{10}$ are constants, $m(\geqslant 0)$ and $k(\geqslant 1)$ are integers. Expressions like (A 19) are called symmetry-adapted functions. The projection of $c2mm(b)$ onto Γ_2 is zero, as would be expected. The same is true of ρ^3_{22} and ρ^4_{22}, because the plane group does not contain a black-and-white generator C'_{2z}. If the transition was to $cm(b)$ these two terms would not be zero.

The planforms described by (A 19) are best illustrated by displacing the positions of the minima of T_0 by $\delta\nabla T_1$, where δ is an arbitrary constant. Figure 10(*b*) was constructed in this way and transforms like Γ_3.

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